

Ramsey Rule with Progressive Utility in Long Term Yield Curves Modeling*

El Karoui Nicole,[†] Hillairet Caroline,[‡] Mrad Mohamed[§]

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Abstract

The purpose of this paper relies on the study of long term yield curves modeling. Inspired by the economic litterature, it provides a financial interpretation of the Ramsey rule that links discount rate and marginal utility of aggregate optimal consumption. For such a long maturity modelization, the possibility of adjusting preferences to new economic information is crucial. Thus, after recalling some important properties on progressive utility, this paper first provides an extension of the notion of a consistent progressive utility to a consistent pair of progressive utilities of investment and consumption. An optimality condition is that the utility from the wealth satisfies a second order SPDE of HJB type involving the Fenchel-Legendre transform of the utility from consumption. This SPDE is solved in order to give a full characterization of this class of consistent progressive pair of utilities. An application of this results is to revisit the classical backward optimization problem in the light of progressive utility theory, emphasizing intertemporal-consistency issue. Then we study the dynamics of the marginal utility yield curve, and give example with backward and progressive power utilities.

Keywords: Market-consistent progressive utility of investment and consumption, Stochastic partial differential equations, Intertemporal-consistency, Forward/backward portfolio optimization, Ramsey rule, Yields curves.

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[†]LPMA, UMR CNRS 6632, Université Pierre et Marie Curie, CMAP, UMR CNRS 7641, École Polytechnique

[‡]CMAP, UMR CNRS 7641, École Polytechnique,

[§]LAGA, UMR CNRS 7539, Université Paris 13

Introduction

This paper focuses on the modelization of long term yield curves. For the financing of ecological project, for the pricing of longevity-linked securities or any other investment with long term impact, modeling long term interest rates is crucial. The answer cannot be found in financial market since for longer maturities, the bond market is highly illiquid and standard financial interest rates models cannot be easily extended. Nevertheless, an abundant literature on the economic aspects of long-term policy-making (i.e. a time horizon between 50 to 200 years), has been developed. The issue is addressed at a macroeconomic level, where long run interest rates have not necessarily the same meaning than in financial market. The Ramsey rule, introduced by Ramsey in his seminal work [35] and further discussed by numerous economists such as Gollier [9, 13, 8, 12, 11, 7, 10] and Weitzman [39, 40], is the reference equation to compute discount rate, that allows to evaluate the future value of an investment by giving a current equivalent value. The Ramsey rule links the discount rate with the marginal utility of aggregate consumption at the economic equilibrium. Even if this rule is very simple, there is no consensus among economists about the parameters that should be considered, leading to very different discount rates. But economists agree on the necessity of a sequential decision scheme that allows to revise the first decisions in the light of new knowledge and direct experiences: the utility criterion must be adaptative and adjusted to the information flow. In the classical optimization point of view, this adaptative criteria is called consistency. In that sense, market-consistent progressive utilities, studied in El Karoui and Mrad [21, 20], are the appropriate tools to study long term yield curves.

Indeed, in a dynamic and stochastic environment, the classical notion of utility function is not flexible enough to help us to make good choices in the long run. M. Musiela and T. Zariphopoulou [29, 28, 27] were the first to suggest to use instead of the classical criterion the concept of progressive dynamic utility, that gives an adaptative way to model possible changes over the time of individual preferences of an agent. Obviously the dynamic utility must be consistent with respect to a given investment universe; this question has been studied from a PDE point of view in [21]. Motivated by the Ramsey rule (in which the consumption rate is a key process), we extend the notion of market-consistent progressive utility with consumption: the agent invest in a financial market and consumes a part of her wealth at each instant. This progressive utilities of investment and consumption were considered at first by Berrier and Tehranchi [1] in the particular case of a zero volatility. This paper studies the general case with a different approach.

In a financial framework, it is natural to link yield curves and zero-coupon, whose pricing in incomplete market is a complex question. Utility functions are also the cornerstone in the utility indifference pricing method, for the pricing of non-replicable contingent claim. For a small amount of transaction, this pricing method leads to a linear pricing rule (see [4]) called the Davis price or marginal utility price. As the zero-coupon bond market is highly illiquid for long maturity, it is relevant to study utility indifference pricing

method for progressive utility with consumption. This paper also points out the similarities and the differences between progressive utilities and the value function of backward classical utility maximization problem. Although the backward classical value function is a progressive utility (cf Mania and Tevzadze [26] for the case without consumption), the way the classical optimization problem is posed is very different from the progressive utility problem. In the classical approach, the optimal processes are computed through a backward analysis, emphasizing their dependency on the horizon of the optimization problem, and leading to intertemporality issues. In the progressive approach, we propose regularity conditions on the utilities characteristics that ensure the existence of consistent utilities and of optimal solutions.

We illustrate those issues on the example of long term discount rate and yield curves. According to the Ramsey rule, we show that equilibrium interest rate and marginal utility interest rate coincide, being careful that this last curve is robust only for small trades. For replicable bonds, equilibrium interest rate and market interest rate are the same. Finally, we study the dynamics of the marginal utility yield curve, in the framework of progressive and backward power utilities (since power utilities are the most commonly used in the economic literature). Special attention is paid on the impact on the yield curves of the maturity of the underlying optimization problem.

The paper is organized as follows, with a concern for finding a workable accommodation between intuition and technical results. For more technical details, the interested reader may refer to [21]. Section 2 starts with the definition of Itô progressive utilities and characterizes these concave Itô's random fields as primitives of SDEs. A special attention is paid to the dynamics of the Fenchel conjugate utility random field, yielding to a very intuitive SPE for the marginal conjugate utility. Section 3 is a technical section where as in H.Kunita [15], "Sobolev spaces" of processes are introduced, in order to study rigorously the properties of monotonicity, differentiability and concavity, both for random fields and solutions of SDEs. Then, the link between non linear SPDE and SDE is detailed, providing a path representation of solution of SDEs.

Section 4 introduces the investment universe and studies market-consistent progressive utilities of investment and consumption. From consistency property we derive a SPDE of HJB type satisfied by the dynamic utility of investment and consumption. Based on the connection between SDEs and SPDEs developed in Section 3 and using same stochastic flows technics as in [21], a closed formula for these forward consistency utilities is given, in term of the inverse flow of the optimal wealth. Special attention is paid to the example of power consistent utility. This section ends with some results on marginal utility indifference pricing, as an application of utility maximization.

Application to yield curve dynamics is given in Section 5. After introducing the economic framework for the computation of long term discount rates, we give a financial interpretation of the Ramsey rule and we study the dynamics of the marginal utility yield curve. More precise properties of the yield curve are given in the framework of power

utilities and log-normal market, in particular on the impact of the terminal horizon.

1 Progressive Utility

Motivated by the necessity of more flexible criterium with respect to the uncertainty of the universe, we introduce the notion of progressive utility. All stochastic processes are defined on a standard filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is assumed to be right continuous and complete. The probability measure \mathbb{P} is a reference probability, often the historical probability.

Progressive utility and its Fenchel conjugate We start with the definition of a progressive utility as progressive random field on \mathbb{R}_+^* concave and increasing with respect to the parameter. Given its importance in convex analysis, we introduce together its convex conjugate $\tilde{\mathbf{U}}$ (also called conjugate progressive utility (CPU)).

Definition 1.1 (Progressive Utility).

- (i) a) A progressive utility is a continuous progressive random field on \mathbb{R}_+^* , $\mathbf{U} = \{U(t, x); t \geq 0, x > 0\}$ such that, for every (t, ω) , $x \mapsto U(\omega, t, x)$ is a strictly concave, strictly increasing, and non negative utility function.
- b) INADA CONDITION \mathbf{U} is assumed to be \mathcal{C}^2 -random field, satisfying Inada conditions: for every (t, ω) , $U(t, \omega, x)$ goes to 0 when x goes to 0 and the derivative $U_x(t, \omega, x)$ (also called marginal utility) decreases from ∞ to 0.
- (ii) The progressive convex conjugate (also called Fenchel conjugate) of the progressive utility \mathbf{U} is the progressive random field $\tilde{\mathbf{U}}$ defined on \mathbb{R}_+^* by

$$\tilde{\mathbf{U}} = \{\tilde{U}(t, y); t \geq 0, y > 0\}, \text{ where } \tilde{U}(t, y) \stackrel{\text{def}}{=} \max_{x > 0, x \in Q^+} (U(t, x) - xy).$$

Under Inada condition, $\tilde{\mathbf{U}}$ is twice continuously differentiable, strictly convex, strictly decreasing, with $\tilde{U}(., 0^+) = U(+\infty)$, $\tilde{U}(., +\infty) = U(0^+)$, a.s.

- (iii) The marginal utility random field \mathbf{U}_x is the inverse of the opposite of the marginal conjugate utility random field $\tilde{\mathbf{U}}_y$, that is $U_x(t, .)^{-1}(y) = -\tilde{U}_y(t, y)$, with $\tilde{U}_y(., 0^+) = -\infty$, $\tilde{U}_y(., +\infty) = 0$, under Inada condition.

- (iv) The bi-dual relation holds true $U(t, x) = \inf_{y > 0, y \in Q^+} (\tilde{U}(t, y) + xy)$.
Moreover $\tilde{U}(t, y) = U(t, -\tilde{U}_y(t, y)) + \tilde{U}_y(t, y)y$, and $U(t, x) = \tilde{U}(t, U_x(t, x)) + x U_x(t, x)$.

Progressive utility is an example of stochastic process depending on a real parameter x , also called *progressive random field* \mathbf{X} . It is useful to specify in some sense some properties that have to be considered when this additional parameter x is taken into account. In particular, we say that the random field $\mathbf{X} \in \mathcal{F}_\infty \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}^+)$ satisfies a property \mathcal{P} , if there exists $N \in \mathcal{F}_\infty$ with $\mathbb{P}(N) = 0$, such that the property is satisfied on N^c . For instance, a random field X is said to be progressive, (predictable, optional) if there exists $N \in \mathcal{F}_\infty$ such that for every $\omega \in N^c$, for every $x \in \mathbb{R}_+$, the process $t \mapsto X_t(\omega, x)$

is progressively measurable. Another family of examples is given by properties relative to the parameter x : for any $\omega \in N^c$, for every $t > 0$, $x \mapsto X(t, x)(\omega)$ satisfies the property \mathcal{P} . In particular, all previous properties as concavity, derivability and so on, may be understood in this sense. The symbol \mathbb{P} *a.s.* is used to say that the negligible set is not depending on x .

To highlight the intuition, Section 2 presents the key ideas that will guide us throughout the rest of this work, with little regard to the assumptions. Section 3 completes then the study by focusing on the conditions under which our assumptions are satisfied.

2 Itô's Progressive Utility

This section uses tools developed in [21] and recalls some important results on Itô's progressive utility that will be useful for this work.

2.1 Itô's progressive utility and SDE

We focus on continuous progressive utilities \mathbf{U} which are a collection of Itô's semimartingales depending on a parameter driven by a n -dimensional Brownian motion $W = (W^1, \dots, W^n)$ defined on the probability space $(\Omega, \mathbb{F}, \mathbb{P})$. From H.Kunita [15], there exist two progressive random fields $(\beta(t, x), \gamma(t, x))$, called *local characteristics* of \mathbf{U} so that \mathbb{P} - *a.s.*,

$$dU(t, x) = \beta(t, x)dt + \gamma(t, x).dW_t \quad (2.1)$$

As usual, the random field β is called the drift characteristic, and the random field γ is called the diffusion characteristic. For $t = 0$, the deterministic utilities $U(0, \cdot)$ and $V(0, \cdot)$ are denoted $u(\cdot)$ and $v(\cdot)$ and in the following small letters u and v design deterministic utilities while capital letters refer to progressive utilities.

A first step is to give conditions on the local characteristics (β, γ) such that the progressive random field \mathbf{U} defined by (2.1) is a progressive utility, that is monotonic and concave with respect to x . It is often easier to prove that the progressive marginal utility \mathbf{U}_x is strictly decreasing and strictly positive, with range $(0, \infty)$.

Proposition 2.1. (i) *We assume \mathbf{U} is regular enough, so that the first and second derivative random fields \mathbf{U}_x and \mathbf{U}_{xx} are also Itô's random fields, with local characteristics (β_x, γ_x) , and $(\beta_{xx}, \gamma_{xx})$. We recall that $-\mathbf{U}_x$ is equal to the derivative of the conjugate utility \tilde{U}_y .*

(ii) **INTRINSIC SDE** *The marginal stochastic utility \mathbf{U}_x (up to the change of initial condition $x = -\tilde{u}_y(z)$) is a strong solution $Z(\cdot) = U_x(\cdot, -\tilde{u}_y(z))$ of the following one dimensional stochastic differential equation $SDE(\mu, \sigma)$, that is \mathbb{P} *a.s.*,*

$$\begin{cases} dZ_t &= \mu(t, Z_t)dt + \sigma(t, Z_t) dW_t, & Z_0 = z \\ \mu(t, z) &:= \beta_x(t, -\tilde{U}_y(t, z)), & \mu(t, 0) = 0 \\ \sigma(t, z) &:= \gamma_x(t, -\tilde{U}_y(t, z)), & \sigma(t, 0) = 0 \end{cases} \quad (2.2)$$

The solution Z is monotonic with respect to its initial condition, with range $(0, \infty)$.

(iii) **STOCHASTIC UTILITY CHARACTERIZATION AS PRIMITIVE OF SDE** Let consider a $SDE(\hat{\mu}, \hat{\sigma})$, $dZ_t = \hat{\mu}(t, Z_t)dt + \hat{\sigma}(t, Z_t)dW_t$, $Z_0 = z$ and assume the existence of a strong global solution $Z(z)$, increasing and differentiable in z with range $(0, \infty)$. Then, for any utility function u such that $Z(u_x(x))$ is Lebesgue-integrable in a neighborhood of $x = 0$, the primitive $\mathbf{U} = \{U(t, x) = \int_0^x Z_t(u_x(z))dz, t \geq 0, x > 0\}$ is a progressive utility.

Comment (i) The **existence** of strong global solution of $SDE(\mu, \sigma)$ is proved by using the same argument than in the deterministic case, when the coefficients are uniformly Lipschitz, with (random) time depending Lipschitz bound, (Protter [32], or for more exhaustive study, see Kunita [15]). A constant Lipschitz bound C corresponds to the classical framework of Lipschitz SDE, and the range property is well-known.

(ii) The notion of "global solution" expresses that the solution $(Z_t(z))$ exists for all $t \geq 0$. Under weaker assumptions, the solution may be defined only up to a finite lifetime $\zeta(z)$, before exploding. More details will be given in the next section.

(iii) **Sufficient conditions** on local characteristics (β, γ) of an Itô's random field \mathbf{U} to be a progressive utility may be exhibited: in particular, if there exist random Lipschitz bounds C_t^i and K_t^i with $\int_0^T C_t^i dt < +\infty$ and $\int_0^T |K_t^i|^2 dt < +\infty$ for any T , such that $\mathbb{P}a.s.$,

$$\begin{cases} |\beta_x(t, x)| \leq C_t |U_x(t, x)|, & \|\gamma_x(t, x)\| \leq K_t |U_x(t, x)| \\ |\beta_{xx}(t, x)| \leq C_t^1 |U_{xx}(t, x)|, & \|\gamma_{xx}(t, x)\| \leq K_t^1 |U_{xx}(t, x)| \end{cases} \quad (2.3)$$

The coefficients of the intrinsic $SDE(\mu, \sigma)$ are uniformly Lipschitz and \mathbf{U} is a progressive utility.

2.2 Dynamics of Convex Conjugate Progressive Utility

The study of the convex conjugate $\tilde{\mathbf{U}}$ of a progressive utility \mathbf{U} is based on the well-known identity (Definition 1.1) $\tilde{U}(t, y) = U(t, -\tilde{U}_y(t, y)) + y\tilde{U}_y(t, y)$, and request to know the dynamics of the \mathcal{C}^2 -semimartingale $U(t, x)$ along the process $-\tilde{U}_y(t, y)$. Calculations are based on Itô-Ventzel's formula, an extension of the classical Itô formula. We refer to Ventzel [38] and Kunita [15] (Theorem 3.3.1) for different variants of this formula.

Proposition 2.2 (Itô-Ventzel's Formula). *Consider a \mathcal{C}^2 -Itô semimartingale \mathbf{F} with local characteristics (ϕ, ψ) , such that \mathbf{F}_x is also an Itô semimartingale, with characteristics (ϕ_x, ψ_x) . For any continuous Itô semimartingale X , $F(., X)$ is an Itô semimartingale,*

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \phi(s, X_s)ds + \int_0^t \psi(s, X_s).dW_s \\ &+ \int_0^t F_x(s, X_s)dX_s + \frac{1}{2} \int_0^t F_{xx}(s, X_s)\langle dX_s \rangle + \int_0^t \langle dF_x(s, x), dX_s \rangle|_{x=X_s} \end{aligned} \quad (2.4)$$

Comment The first line of the right hand side of the equation corresponds to the dynamics of the process $(F(t, x))_{t \geq 0}$ taken on $(X_t)_{t \geq 0}$, when in the second line, the first

two terms come from the classical Itô's formula. The last term represents the quadratic covariation between $dF_x(t, x)$ and dX_t , at $x = X_t$, which can be written as $\psi_x(t, X_t) \cdot \sigma_t^X dt$ when the diffusion coefficient of X is the vector σ_t^X .

Itô-Ventzel's formula and monotonic change of variable will help us to establish the relationship between local characteristics of the random fields \mathbf{U} and $\tilde{\mathbf{U}}$.

Theorem 2.3. *Let \mathbf{U} a progressive utility and $\tilde{\mathbf{U}}$ its progressive convex conjugate utility assumed to be \mathcal{C}^2 -Itô's semimartingales with local characteristics (β, γ) and $(\tilde{\beta}, \tilde{\gamma})$. We also assume that their marginal utilities \mathbf{U}_x , and $\tilde{\mathbf{U}}_y$ are Itô's semimartingales with local characteristics (β_x, γ_x) and $(\tilde{\beta}_y, \tilde{\gamma}_y)$.*

(i) *The dynamics of $\tilde{\mathbf{U}}$ is driven by the non linear second order SPDE,*

$$d\tilde{U}(t, y) = \gamma(t, -\tilde{U}_y(t, y)) \cdot dW_t + \beta(t, -\tilde{U}_y(t, y))dt + \frac{1}{2} \tilde{U}_{yy}(t, y) \|\gamma_x(t, -\tilde{U}_y(t, y))\|^2 dt. \quad (2.5)$$

(ii) *Assume (μ, σ) (the random coefficients of the SDE associated with \mathbf{U}_x) to be fairly regular for the adjoint elliptic operator in divergence form is well defined,*

$$\hat{L}_{t,y}^{\sigma,\mu}(f) = \frac{1}{2} \partial_y (\|\sigma(t, y)\|^2 \partial_y f(t, y)) - \mu(t, y) \partial_y f(t, y). \quad (2.6)$$

Then the marginal conjugate utility $\tilde{\mathbf{U}}_y$ is a monotonic solution of the forward SPDE

$$d\tilde{U}_y(t, y) = -\partial_y(\tilde{U}_y(t, y)\sigma(t, y)) \cdot dW_t + \hat{L}_{t,y}^{\sigma,\mu}(\tilde{U}_y)dt, \quad \tilde{U}_y(0, y) = \tilde{u}_y(y). \quad (2.7)$$

Observe that the derivability of the local characteristics $(\tilde{\beta}, \tilde{\gamma})$ of $\tilde{\mathbf{U}}$ requires the existence of a third derivative for $\tilde{\mathbf{U}}$, and thus for \mathbf{U} . Remark also that (ii) characterizes the inverse of a SDE.

PROOF. Let apply Itô-Ventzel's formula to the regular random field $F(t, x) = U(t, x) - yx$ and to the semimartingale $X_t = -\tilde{U}_y(t, y)$. The following identities will be useful, $F(t, -\tilde{U}_y(t, y)) = \tilde{U}(t, y)$, $U_{xx}(t, -\tilde{U}_y(t, y)) = -1/\tilde{U}_{yy}(t, y)$.

(i) a) Observe that $F_x(t, -\tilde{U}_y(t, y)) = U_x(-\tilde{U}_y(t, y)) - y \equiv 0$, so that the term in $F_x(s, X_s)dX_s$ disappears in the Itô-Ventzel formula; then the diffusion random field $\tilde{\gamma}$ of $\tilde{\mathbf{U}}$ is $\tilde{\gamma}(t, y) = \gamma(t, -\tilde{U}_y(t, y))$. Its derivative $\tilde{\gamma}_y(t, y) = -\gamma_x(t, -\tilde{U}_y(t, y))\tilde{U}_{yy}(t, y)$ is by assumption the diffusion characteristic of $\tilde{\mathbf{U}}_y$. Hence the covariation term is driven by $\langle dF_x(t, x), -d\tilde{U}_y(t, y) \rangle = -\langle \gamma_x(t, x), \tilde{\gamma}_y(t, y) \rangle dt$.

b) The Itô-Ventzel's formula is then reduced to,

$$\begin{aligned} d\tilde{U}(t, y) &= \beta(t, -\tilde{U}_y(t, y))dt - \gamma(t, -\tilde{U}_y(t, y)) \cdot dW_t \\ &= \frac{1}{2} U_{xx}(t, -\tilde{U}_y(t, y)) \langle d\tilde{U}_y(t, y) \rangle - \langle \gamma_x(t, -\tilde{U}_y(t, y)), \tilde{\gamma}_y(t, y) \rangle dt \\ &= \frac{1}{2} U_{xx}(t, -\tilde{U}_y(t, y)) \|\tilde{\gamma}_y(t, y)\|^2 dt - U_{xx}(t, -\tilde{U}_y(t, y)) \|\tilde{\gamma}_y(t, y)\|^2 dt \\ U_{xx}\text{-formulation} &= -\frac{1}{2} U_{xx}(t, -\tilde{U}_y(t, y)) \|\tilde{\gamma}_y(t, y)\|^2 dt \\ \tilde{U}_{yy}\text{-formulation} &= \frac{1}{2} \tilde{U}_{yy}(t, y) \|\gamma_x(t, -\tilde{U}_y(t, y))\|^2 dt. \end{aligned}$$

(ii) The dynamics of $\tilde{\mathbf{U}}_y$ is obtained (by assumption and Theorem 3.1) by differentiating term by term in the previous equation. The use of coefficients $\sigma(t, y) = \gamma_x(t, -\tilde{U}_y(t, y))$ and $\mu(t, y) = \beta_x(t, -\tilde{U}_y(t, y))$ of the SDE associated with \mathbf{U}_x allows us to express $\tilde{\mathbf{U}}_y$ as the solution of a SPDE driven by the operator $\hat{L}_{t,y}^{\sigma,\mu} = \frac{1}{2}\partial_y(\|\sigma(t, y)\|^2\partial_y) - \mu(t, y)\partial_y$,

$$\begin{aligned} d\tilde{U}_y(t, y) &= -\tilde{U}_{yy}(t, y)[\mu(t, y)dt + \sigma(t, y).dW_t] + \partial_y\left(\frac{1}{2}\tilde{U}_{yy}(t, y)\|\sigma(t, y)\|^2\right)dt \\ &= -\partial_y\tilde{U}_y(t, y)\sigma(t, y).dW_t + \hat{L}_{t,y}^{\sigma,\mu}(\tilde{U}_y)dt \end{aligned}$$

The proof is achieved. ■

Remark 2.1. Obviously, we are also interested in the properties of the SDE $(\tilde{\mu}, \tilde{\sigma})$ associated with the monotonic random field $\tilde{\mathbf{U}}_y$, $\tilde{\mu}(t, z) = \tilde{\beta}_y(t, (\tilde{U}_y)^{-1}(t, z))$ and $\tilde{\sigma}(t, z) = \tilde{\gamma}_y(t, (\tilde{U}_y)^{-1}(t, z))$. Given that $(\tilde{U}_y)^{-1}(t, z) = -U_x(t, z)$,

$$U_{xx}(t, z)\tilde{\sigma}(t, z) = \gamma_x(t, -z) \text{ and } U_{xx}(t, -z)\tilde{\mu}(t, -z) = \left(\beta_x(t, z) - \frac{1}{2}\partial_x\left(\frac{\|\gamma_x(t, z)\|^2}{U_{xx}(t, z)}\right)\right).$$

It is clear that these coefficients are not globally Lipschitz. The problem in studying directly the SDE $(\tilde{\sigma}, \tilde{\mu})$ is the existence of a possible explosion time $\tau(x)$ as it is shown in the next section 3.1, Theorem 3.1. Let us first introduce some additional tools about regularity issues.

3 Regularity of Itô's random fields and SPDEs

In this section, we focus on the regularity of the local characteristics of Itô's random fields in order to justify flows properties, in particular in terms of derivatives, monotony...I We also establish a connection between SDEs and SPDEs, that will be useful to characterize the market-consistent progressive utilities from their dynamics.

3.1 Regularity issues

We shall discuss the regularity of an Itô semimartingale random field $F(t, x) = F(0, x) + \int_0^t \phi(s, x)ds + \int_0^t \psi(s, x).dW_s$ in connection with the regularity of its local characteristics (ϕ, ψ) and conversely. We are also concerned with the same questions concerning SDEs solutions, where the spatial parameter is the initial condition. As in the deterministic case, it is necessary to introduce some spatial norms very similar to Sobolev norms.

Definition of norms and spaces Let ϕ be a continuous \mathbb{R}^k -valued progressive random field and let m be a non-negative integer, and δ a number in $(0, 1]$. We need to control the asymptotic behavior in 0 and ∞ of ϕ , and the regularity of its Hölder derivatives when there exist. More precisely, let ϕ be in the class $\mathcal{C}^{m,\delta}(]0, +\infty[)$, i.e. (m, δ) -times continuously differentiable in x for any t , a.s.

(i) For any subset $K \subset]0, +\infty[$, we define the family of random (Hölder) K -semi-norms

$$\begin{cases} \|\phi\|_{m,K}(t, \omega) = \sup_{x \in K} \frac{\|\phi(t, x, \omega)\|}{x} + \sum_{1 \leq j \leq m} \sup_{x \in K} \|\partial_x^j \phi(t, x, \omega)\| \\ \|\phi\|_{m,\delta,K}(t, \omega) = \|\phi\|_{m,K}(t, \omega) + \sup_{x, y \in K} \frac{\|\partial_x^m \phi(t, x, \omega) - \partial_x^m \phi(t, y, \omega)\|}{|x - y|^\delta}. \end{cases} \quad (3.1)$$

The case $(m = 0, \delta = 1)$ corresponds to the local version of the Lipschitz case used in Section 1. When K is all the domain $]0, +\infty[$, we simply write $\|\cdot\|_m(t, \omega)$, or $\|\cdot\|_{m,\delta}(t, \omega)$.

(ii) The previous semi-norms are related to the spatial parameter. We add the temporal dimension in assuming these semi-norms (or the square of the semi-norm) to be integrable in time with respect to the Lebesgue measure on $[0, T]$ for all T . Then, as in Lebesgue's Theorem, we can differentiate, pass to the limit, commute limit and integral for the random fields. Calligraphic notation recalls that these semi-norms are random.

a) $\mathcal{K}_{loc}^{m,\delta}$ (resp. $\overline{\mathcal{K}}_{loc}^{m,\delta}$) denotes the set of all $\mathcal{C}^{m,\delta}$ -random fields such that for any compact $K \subset]0, +\infty[$, and any T , $\int_0^T \|\phi\|_{m,\delta;K}(t, \omega) dt < \infty$, (resp. $\int_0^T \|\psi\|_{m,\delta;K}^2(t, \omega) dt < \infty$).

b) When these different norms are well-defined on the whole space $]0, +\infty[$, the derivatives (up to a certain order) are bounded in the spatial parameter, with integrable (resp. square integrable) in time random bound. In this case, we use the notations $\mathcal{K}_b^m, \overline{\mathcal{K}}_b^m$ or $\mathcal{K}_b^{m,\delta}, \overline{\mathcal{K}}_b^{m,\delta}$.

Regularity properties of random fields and SDEs The following proposition is a short presentation of technical results in Kunita [15].

Proposition 3.1 (Differential rules for random fields). *Let \mathbf{F} be an Itô semimartingale random field with local characteristics (ϕ, ψ) , $F(t, x) = F(0, x) + \int_0^t \phi(s, x) ds + \int_0^t \psi(s, x) dW_s$*

- (i) *If \mathbf{F} is a $\mathcal{K}_{loc}^{m,\delta}$ -semimartingale for some $m \geq 0$, $\delta \in (0, 1]$, its local characteristics (ϕ, ψ) are of class $\mathcal{K}_{loc}^{m,\varepsilon} \times \overline{\mathcal{K}}_{loc}^{m,\varepsilon}$ for any $\varepsilon < \delta$.*
- (ii) *Conversely, if the local characteristics (ϕ, ψ) are of class $\mathcal{K}_{loc}^{m,\delta} \times \overline{\mathcal{K}}_{loc}^{m,\delta}$, then \mathbf{F} is a $\mathcal{K}_{loc}^{m,\varepsilon}$ -semimartingale for any $\varepsilon < \delta$.*
- (iii) *In any cases, for $m \geq 1, \delta \in (0, 1]$, the derivative random field \mathbf{F}_x is an Itô random field with local characteristics (ϕ_x, ψ_x) .*

The particular case of SDEs solutions is of major interest for the applications. The presentation follows [21].

Theorem 3.2 (Flows property of SDE). a) **Strong solution** *Consider a SDE (μ, σ) , with uniformly Lipschitz coefficients $(\mu, \sigma) \in \mathcal{K}_b^{0,1} \times \overline{\mathcal{K}}_b^{0,1}$. There exists a unique strong solution X such that*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x.$$

- (i) *If $\mu \in \mathcal{K}_b^{m,\delta}$ and $\sigma \in \overline{\mathcal{K}}_b^{m,\delta}$ for some $m \geq 1, \delta \in (0, 1]$, the solution $\mathbf{X} = (X_t^x, x > 0)$ is a $\mathcal{K}_{loc}^{m,\varepsilon}$ semimartingale for any $\varepsilon < \delta$. The inverse \mathbf{X}^{-1} of \mathbf{X} is also of class \mathcal{C}^m . Then, the derivatives \mathbf{X}_x and $1/\mathbf{X}_x$ are $\mathcal{K}_{loc}^{m-1,\varepsilon}$ -semimartingales.*
 - (ii) *The local characteristics of \mathbf{X} , $\lambda^X(t, x) = \mu(t, X_t^x)$ and $\theta^X(t, x) = \sigma(t, X_t^x)$ have only local properties and belong to $\mathcal{K}_{loc}^{m,\varepsilon} \times \overline{\mathcal{K}}_{loc}^{m,\varepsilon}$ for any $\varepsilon < \delta$.*
- b) **Local SDEs** *Assume only local property on the coefficients, $(\mu, \sigma) \in \mathcal{K}_{loc}^{0,1} \times \overline{\mathcal{K}}_{loc}^{0,1}$.*
- (i) *Then, for any initial condition x , the SDE has a unique maximal monotonic solution*

(X_t^x) up to an explosion time $\tau(x)$, and (X_t^x) is a global solution if and only if the explosion time $\tau(x)$ is equal to ∞ for all $x > 0$ a.s..

(ii) If $(\mu, \sigma) \in \mathcal{K}_{loc}^{m,\delta} \times \overline{\mathcal{K}}_{loc}^{m,\delta}$, $m \geq 1$, $0 < \delta \leq 1$, $X_t(\cdot)$ is of class $\mathcal{C}^{m,\varepsilon}$, $\varepsilon < \delta$ on $\{\tau(x) > t\}$.

3.2 Solvable SPDEs via SDEs

Since we are only concerned with non explosive solution to SDEs, we give a name to this specific class.

Class $\mathcal{S}^{m,\delta}$: A SDE (μ, σ) with $(\mu, \sigma) \in \mathcal{K}_{loc}^{m,\delta} \times \overline{\mathcal{K}}_{loc}^{m,\delta}$ whose local solution is non explosive is said to be of **class $\mathcal{S}^{m,\delta}$** .

The typical example of SDE in $\mathcal{S}^{m,\delta}$ is the SDE associated with the marginal conjugate utility considered as the inverse of a solution of SDE (μ, σ) as in Theorem 2.3 for which a SPDE has been associated in a very natural way in Theorem 2.3, under the the assumption that the inverse flow $-\tilde{\mathbf{U}}_y$ of \mathbf{U}_x is a semimartingale. This may seem obvious, but generally the inverse of a semimartingale is not necessarily a semimartingale. A way to define the regularity required on the coefficients (μ, σ) is to formally transform the SPDE into a SDE and to apply previous result on SDE. We also point that the inverse flow is less regular than the flow itself. The SPDE point of view is more efficient to calculate the stochastic transformation of the solution or of its inverse, and allows us to establish an exact connection between SDEs and SPDEs. This last point of view is well-suited to the study of progressive utilities developed in this paper.

Proposition 3.3. *Let $(X(t, x))$ be the monotonic solution of a SDE (μ, σ) of class $\mathcal{S}^{m,\delta}$, $m \geq 2$, $\delta \in]0, 1]$, so that as random field $(X(t, x))$ and its local characteristics $(\lambda(t, x) = \mu(t, X(t, x)))$ and $(\theta(t, x) = \sigma(t, X(t, x)))$ are of class $\mathcal{K}_{loc}^{m,\varepsilon}$ and $\mathcal{L}_{loc}^{m,\varepsilon} \times \overline{\mathcal{K}}_{loc}^{m,\varepsilon}$ for any $0 < \varepsilon < 1$. We are concerned with the SDE $(\tilde{\mu}, \tilde{\sigma})$*

$$d\xi_t = -\frac{1}{X_x(t, \xi_t)} \left[\left(\lambda(t, \xi_t) - \frac{1}{2} \partial_x \left(\frac{\|\theta\|^2}{X_x} \right) (t, \xi_t) \right) dt + \theta(t, \xi_t) . dW_t \right], \quad \xi_0 = z, \quad (3.2)$$

where $\tilde{\sigma}(t, z) = -\frac{\theta(t, z)}{X_x(t, z)}$ and $\tilde{\mu}(t, z) = \frac{1}{X_x(t, z)} \left(\frac{1}{2} \partial_x \left(\frac{\|\theta\|^2}{X_x} \right) (t, z) - \lambda(t, z) \right)$.

(i) The SDE $(\tilde{\mu}, \tilde{\sigma})$ is of class $\mathcal{S}^{m-2,\varepsilon}$ ($0 < \varepsilon < \delta$) and its unique monotonic solution ξ^z is the inverse flow X^{-1} of X .

(ii) Consequently, the inverse X^{-1} of X is a semimartingale and belongs to the class $\mathcal{K}_{loc}^{m-2,\varepsilon} \cap \mathcal{C}^m$.

PROOF. (i) According to Theorem 2.3, \mathbf{X} may be considered up to a change of initial variable as a marginal progressive utility \mathbf{U}_x . From Remark 2.1, if its inverse ξ^X is "regular", then ξ^X is solution of SDE $(\tilde{\mu}, \tilde{\sigma})$ with $X_x(t, z) \tilde{\sigma}(t, z) = \sigma_x(t, X(t, z))$ and

$$X_x(t, z) \tilde{\mu}(t, z) = \frac{1}{2} \partial_x \left(\frac{\|\sigma(t, X(t, x))\|^2}{X_x(t, x)} \right) (t, z) - \mu(t, X(t, z))$$

The coefficients of the local SDE $(\tilde{\mu}, \tilde{\sigma})$ are of class $\mathcal{K}_{loc}^{m-2,\varepsilon} \times \overline{\mathcal{K}}_{loc}^{m-1,\varepsilon}$. Then, the SDE has a unique maximal solution $\xi(t, z)$ up to a life time $\tau(z)$. It remains to show that by

the Itô-Ventzel formula $X(t, \xi(t, z)) = z$ on $[0, \tau(z))$. Assume this is proven. Then the continuous (in time) process $X(t, \xi_t^z)$ is constant a.s. on $[0, \tau(z))$. At time $t = \tau(z) < \infty$, $\xi(t, z) = \infty$ and $X(t, \infty) = \infty$. On the other hand, by continuity, $X(t, \xi(t, z)) = z$ if $t = \tau(z) < \infty$. To avoid contradiction, necessarily $\tau(z) = \infty$, a.s.. So ξ is the inverse flow $\xi^{\mathbf{X}}$ of \mathbf{X} . The proof of $X(t, \xi(t, z)) = z$ is very similar to the next proof, so we omit it here. ■

We come back now to the SPDE point of view as in Section 2

Theorem 3.4. *Let us consider a SDE (μ, σ) of class $\mathcal{S}^{m, \delta}$ with $m \geq 2, \delta \in (0, 1]$, and its adjoint operator $\widehat{L}_{t,z}^{\sigma, \mu} = \frac{1}{2} \partial_z (\|\sigma(t, z)\|^2 \partial_z) - \mu(t, z) \partial_z$. Denote by X its unique solution.*

(i) *The inverse flow $X^{-1} = \xi^X$ of X is a strictly monotonic solution of class $\mathcal{K}_{loc}^{m-2, \delta} \cap \mathcal{C}^m$ of SPDE $(\widehat{L}^{\sigma, \mu}, -\sigma \partial_z)$, with initial condition $\xi_0(z) = z$,*

$$d\xi(t, z) = -\xi_z(t, z) \sigma(t, z) . dW_t + \widehat{L}_{t,z}^{\sigma, \mu}(\xi) dt. \quad (3.3)$$

(ii) *Conversely, ($m \geq 2$), let ξ be a $\mathcal{K}_{loc}^{1, \delta} \cap \mathcal{C}^2$ -regular solution of SPDE $(\widehat{L}^{\sigma, \mu}, -\sigma \partial_z)$ (3.3). Then, $\xi(t, X(t, x)) \equiv x$ and ξ is the strictly monotonic inverse flow $X^{-1} := \xi^X$ of X . Moreover, uniqueness holds true for the SPDE $(\widehat{L}^{\sigma, \mu}, -\sigma \partial_z)$ in the class of $\mathcal{K}_{loc}^{1, \delta} \cap \mathcal{C}^2$ -regular solutions.*

PROOF. (i) We start with a monotonic solution ξ of class $\mathcal{K}_{loc}^{1, \delta} \cap \mathcal{C}^2$ of the SPDE: $d\xi(t, z) = -\xi_z(t, z) \sigma(t, z) . dW_t + \widehat{L}_{t,z}^{\sigma, \mu}(\xi) dt$.

(ii) From Theorem 3.1, ξ is regular enough to use Itô-Ventzel's formula with the solution $X(t, x) = X_t^x$ of the SDE (μ, σ) to compute the dynamics of $H(t, x) = \xi(t, X(t, x))$. In the next equation, we do not recall the parameter x .

$$\begin{aligned} dH_t &= (-\xi_z(t, X_t) \sigma(t, X_t) - \xi_z(t, X_t) \sigma(t, X_t)) . dW_t \\ &+ \left(\widehat{L}^{\sigma, \mu}(\xi) + \frac{1}{2} \xi_{zz} \|\sigma\|^2 + \mu \xi_z + \partial_z(-\xi_z \sigma) . \sigma \right) (t, X_t) dt \\ &= \left(\xi_{zz} \|\sigma\|^2 + \frac{1}{2} \xi_z (\partial_z \|\sigma\|^2) - \partial_z(\xi_z) \|\sigma\|^2 - \frac{1}{2} \xi_z (\partial_z \|\sigma\|^2) \right) (t, X_t) dt = \mathbf{0} \end{aligned}$$

The random field $H(t, x) = \xi(t, X(t, x))$ is constant in time and equal to its initial condition x . This finishes the proof that X is the inverse flow of ξ . The $\mathcal{S}^{m, \delta}$ -SDE (μ, σ) has only one solution X . Then any "regular" solution ξ of the SPDE is the inverse of X and then is unique. ■

The next result, useful for applications, is a slight extension of the previous one. It establishes a connection between a more general second order SPDE and two SDEs. It is based on the observation that if ξ is the inverse of the monotonic solution X of SDE (μ^X, σ^X) and if $\phi \in \mathcal{C}^2$ a regular monotonic function, the process $X(\cdot, \phi(x))$ satisfies the same SDE (μ^X, σ^X) , and so its inverse $\phi^{-1}(\xi(z))$ satisfies the same SPDE than ξ . The extension describes the SPDEs associated with the compound processes $Y(t, \xi(t, z))$, identified as the unique solution.

Theorem 3.5. *Let X be a solution of $SDE(\mu^X, \sigma^X)$ and ξ a $\mathcal{K}_{loc}^{1,\delta} \cap \mathcal{C}^2$ -regular solution ($\delta > 0$) of the SPDE $(\widehat{L}^X, -\sigma^X \partial_z)$, where $\widehat{L}_{t,z}^X = \frac{1}{2} \partial_z (\|\sigma^X(t, z)\|^2 \partial_z) - \mu^X(t, z) \partial_z$.*

(i) *Let Y be a solution of class $\mathcal{K}_{loc}^{1,\delta} \cap \mathcal{C}^2$ of $SDE(\mu^Y, \sigma^Y)$ and ϕ any function in \mathcal{C}^2 . Then the random field $Y(t, \phi(\xi(t, z))) = G(t, z)$ evolves as,*

$$\begin{aligned} dG(t, z) &= \sigma^Y(t, G(t, z)) \cdot dW_t + \mu^Y(t, G(t, z)) dt \\ &\quad - \partial_z G(t, z) \sigma^X(t, z) [dW_t + \sigma_y^Y(t, G(t, z)) dt] + \widehat{L}_{t,z}^X(G)(t, z) dt \end{aligned} \quad (3.4)$$

with initial condition $G(0, z) = \phi(z)$.

(ii) **Solvable SPDE:** *Conversely, let G be a solution of class $\mathcal{K}_{loc}^{1,\delta} \cap \mathcal{C}^2$ of the SPDE (3.4); then the process $G(t, X_t(z))$ with initial condition $\phi(z) := G(0, z)$ is solution of the $SDE(\mu^Y, \sigma^Y)$. If uniqueness holds true for this equation, then $G(t, z) = Y_t(t, \phi(\xi(t, z)))$ and uniqueness also holds true for the SPDE (3.4).*

Note the different nature of assumptions (which may be equivalent) in the assertions of this theorem. In (i), we assume that the coefficients are regular enough such that Y satisfies the Itô-Ventzel assumptions and such that the inverse ξ of X is an Itô semimartingale, while in (ii) we only suppose the existence of X (without regularity), but in return we assume the existence of a smooth solution G of the SPDE (3.4).

4 Market-Consistent progressive utilities of investment and consumption

The notion of progressive utility is very general and should be specified so as to represent more realistically the dynamic evolution of the individual preferences of an investor in a given financial market. As in statistical learning, the utility criterium is dynamically adjusted to be the best given the market past information. So, the market inputs may be viewed as a calibration universe and gives a test-class of processes on which the utility is chosen to provide the best satisfaction. The market input is described by a vector space \mathcal{X}^c of portfolios and consumption incorporating feasibility and trading constraints and high liquidity.

The existence of an admissible strategy giving the maximal satisfaction to the investor, which will be preserved at all times in the future, explains *the martingale property* in Definition 4.3. On the other hand if the strategy in \mathcal{X}^c fails to be optimal then it is better not to make investment. The optimal strategy may be viewed as a benchmark for the investor using the progressive utility \mathbf{U} . Once his consistent progressive utility is defined, an investor can then turn to a portfolio optimization problem in a larger financial market or to calculate indifference prices. Before extending, in a framework with consumption, the definition of a consistent dynamic utility, introduced in [21, 20, 19] and following Musiela and Zariphopoulou [29, 27], we first define the investment universe and the set of test processes.

4.1 The investment universe with consumption.

We consider an incomplete Itô market, defined on the filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ driven by the n -standard Brownian motion W . As usual, the market is characterized by a short rate (r_t) and a n -dimensional risk premium vector (η_t) .

The agent may invest in this financial market and is allowed to consume a part of his wealth at the progressive rate $c_t \geq 0$. To be short, we give the mathematical definition of the class of admissible strategies (κ_t, c_t) , without specifying the risky assets. The incompleteness of the market is expressed by restrictions on the risky portfolios κ_t constrained to live in a given progressive vector space (\mathcal{R}_t) .

To avoid technicalities, we assume throughout that all the processes satisfy the necessary measurability and integrability conditions such that the following formal manipulations and statements are meaningful.

Definition 4.1 (Test processes). (i) *The self-financing dynamics of a wealth process with risky portfolio κ and consumption rate c is given by*

$$dX_t^{\kappa, c} = X_t^{\kappa, c}[r_t dt + \kappa_t(dW_t + \eta_t dt)] - c_t dt, \quad \kappa_t \in \mathcal{R}_t. \quad (4.1)$$

where c is a positive adapted process, κ is a progressive n -dimensional vector measuring the volatility vector of the wealth $X^{\kappa, c}$, such that $\int_0^T c_t + \|\kappa_t\|^2 dt < \infty, a.s..$

(ii) *A self-financing strategy (κ_t, c_t) is admissible if it is stopped with the bankruptcy of the investor (when the wealth process reaches 0) and if the portfolio κ lives in a given progressive family of vector spaces (\mathcal{R}_t) a.s..*

(iii) *The set of the wealth processes with admissible (κ_t, c_t) , also called test processes, is denoted by \mathcal{X}^c . When portfolios are starting from x at time t , we use the notation $\mathcal{X}_t^c(x)$.*

The following short notations will be used extensively. Let \mathcal{R} be a vector subspace of \mathbb{R}^n . For any $x \in \mathbb{R}^n$, $x^{\mathcal{R}}$ is the orthogonal projection of the vector x onto \mathcal{R} and x^{\perp} is the orthogonal projection onto \mathcal{R}^{\perp} .

The existence of a risk premium η is a possible formulation of the absence of arbitrage opportunity. Since from (4.1), the impact of the risk premium on the wealth dynamics only appears through the term $\kappa_t \cdot \eta_t$ for $\kappa_t \in \mathcal{R}_t$, there is a "minimal" risk premium $(\eta_t^{\mathcal{R}})$, the projection of η_t on the space \mathcal{R}_t ($\kappa_t \cdot \eta_t = \kappa_t \cdot \eta_t^{\mathcal{R}}$), to which we refer in the sequel. Moreover, the existence of $\eta^{\mathcal{R}}$ is not enough to insure the existence of equivalent martingale measure, since in general we do not know if the exponential local martingale $L_t^{\eta^{\mathcal{R}}} = \exp(-\int_0^t \eta_s^{\mathcal{R}} \cdot dW_s - \frac{1}{2} \int_0^t \|\eta_s^{\mathcal{R}}\|^2 ds)$ is a uniformly integrable martingale, density of an equivalent martingale measure. In the following definition, we are interested in the class of the so-called state price density processes Y^{ν} (taking into account the discount factor) who will play the same role for the progressive conjugate utility, than the test processes $X^{\kappa, c}$ for the progressive utility.

Definition 4.2 (State price density process). (i) An Itô semimartingale Y^ν is called a state price density process if for any admissible test process $X^{\kappa,c}$, ($c \geq 0, \kappa \in \mathcal{R}$), $Y^\nu X^{\kappa,c} + \int_0^\cdot Y_s^\nu c_s ds$ is a local martingale. It follows that Y^ν satisfies,

$$dY_t^\nu = Y_t^\nu [-r_t dt + (\nu_t - \eta_t^\kappa) \cdot dW_t], \quad \nu_t \in \mathcal{R}_t^\perp, \quad Y_0^\nu = y \quad (4.2)$$

(ii) Denote \mathcal{Y} the convex family of all state density processes Y^ν where $\nu \in \mathcal{R}^\perp$ and observe that Y^ν is the product of Y^0 ($\nu = 0$) by the density martingale $L_t^\nu = \exp(\int_0^t \nu_s \cdot dW_s - \frac{1}{2} \int_0^t \|\nu_s\|^2 ds)$.

Interesting discussion on the links between these assumptions and the market numeraire $N_t = (Y^0)_t^{-1}$, also called GOP (growth optimal portfolio) can be found in the book by D.Heath & E.Platten [5] and in D.Filipovic & E.Platten [6]. Nevertheless, the use of change of numeraire in our framework is reported in order to limit the size of the paper.

4.2 \mathcal{X}^c -consistent Utility and Portfolio optimization with consumption

As we are interested in optimizing both the terminal wealth and the consumption rate, we introduce two progressive utilities: the first one, \mathbf{U} , for the terminal wealth and the second one, \mathbf{V} , for the consumption rate. From a dynamic point of view, \mathbf{U} and \mathbf{V} will play different roles, only \mathbf{U} will need to be an Itô progressive utility. To express that the adaptative criteria (U, V) are well-adapted to the investment universe, we introduce the following conditions:

Definition 4.3. A \mathcal{X}^c -consistent progressive utility system of investment and consumption is a pair of progressive utilities \mathbf{U} and \mathbf{V} on $\Omega \times [0, +\infty) \times \mathbb{R}^+$ with the following additional properties:

(i) CONSISTENCY WITH THE TEST-CLASS: For any admissible wealth process $X^{\kappa,c} \in \mathcal{X}^c$, and any couple of dates $t < T$,

$$\mathbb{E}(U(T, X_T^{\kappa,c}) + \int_t^T V(s, c_s) ds / \mathcal{F}_t) \leq U(t, X_t^{\kappa,c}), \quad a.s.$$

In other words, the value process $(\mathcal{G}_t^{\kappa,c} = U(t, X_t^{\kappa,c}) + \int_0^t V(s, c_s) ds)$ is a positive supermartingale.

(ii) EXISTENCE OF OPTIMAL STRATEGY: For any initial wealth $x > 0$, there exists an optimal strategy (κ^*, c^*) such that the associated non negative wealth process $X^* = X^{\kappa^*, c^*} \in \mathcal{X}^c$ issued from x satisfies $(\mathcal{G}_t^* = U(t, X_t^*) + \int_0^t V(s, c_s^*) ds)$ is a local martingale.

\mathcal{X}^c -consistent Itô progressive utilities and HJB constraint Theorem 2.1 characterizes Itô progressive utilities in terms of their local characteristics (β, γ) as well as in terms of the parameters (μ, σ) of the intrinsic SDE (2.2) satisfied by U_x . In this

section, we are concerned with the constraints induced on the characteristics (β, γ) of \mathbf{U} by the \mathcal{X}^c consistency property.

The supermartingale/martingale property of processes $\mathcal{G}_t^{\kappa, c}$ implies negative drift for these Itô processes for all $\kappa \in \mathcal{R}, c \geq 0$, and 0 drift for some (κ^*, c^*) . This property yields to Hamilton-Jacobi-Bellman type constraints on the drift $\beta(t, x)$ of \mathbf{U} .

We proceed by verification as usual by introducing a non standard Hamilton-Jacobi-Bellman Stochastic PDE. Observe that the consumption optimization contributes only through the Fenchel-Legendre random field $\tilde{\mathbf{V}}$ of the dynamic utility \mathbf{V} .

Theorem 4.4 (Utility-SPDE). *Let (\mathbf{U}, \mathbf{V}) be a utility system where \mathbf{U} is regular enough to apply Itô's Ventzel formula. Define a monotonic random field $\bar{\zeta}(t, x)$ by $\bar{\zeta}(t, x) = (V_c)^{-1}(t, U_x(t, x)) = -\tilde{V}_y(t, U_x(t, x))$, and a "policy" random field $x\bar{\kappa}_t(x)$ by*

$$x\bar{\kappa}_t(x) = -\frac{1}{U_{xx}(t, x)}(U_x(t, x)\eta_t^{\mathcal{R}} + \gamma_x^{\mathcal{R}}(t, x)). \quad (4.3)$$

Then the \mathcal{X}^c -consistency property of (\mathbf{U}, \mathbf{V}) is implied by the following two assertions:

a) The drift β satisfies the following HJB-constraint

$$\beta(t, x) = -U_x(t, x)xr_t + \frac{1}{2}U_{xx}(t, x)\|x\bar{\kappa}_t(x)\|^2 - \tilde{V}(t, U_x(t, x)). \quad (4.4)$$

b) The SDE $(\bar{\mu}^c, \bar{\sigma})$, with $\bar{\mu}_t^c(x) := r_t x + x\bar{\kappa}_t(x) \cdot \eta_t^{\mathcal{R}} - \bar{\zeta}(t, x)$ and $\bar{\sigma}(t, x) := x\bar{\kappa}_t(x)$ admits a non negative solution $\bar{X} \geq 0$. Furthermore, $(x\bar{\kappa}(x), \bar{c}_t = \bar{\zeta}(t, \bar{X}_t))$ is the optimal strategy of investment and consumption (in general denoted with a $(*)$) with monotonic optimal wealth $X^* = \bar{X}$.

Proof. (i) By Itô-Ventzel's formula (Theorem 2.2), for any test process $X^{\kappa, c}$,

$$\begin{aligned} dU(t, X_t^{\kappa, c}) + V(t, c_t)dt &= \left(U_x(t, X_t^{\kappa, c})X_t^{\kappa, c} \kappa_t + \gamma(t, X_t^{\kappa, c}) \right) \cdot dW_t \\ &+ \left(\beta(t, X_t^{\kappa, c}) + U_x(t, X_t^{\kappa, c})r_t X_t^{\kappa, c} + \frac{1}{2}U_{xx}(t, X_t^{\kappa, c})\mathcal{Q}(t, X_t^{\kappa, c}, \kappa_t) \right) dt \\ &+ (V(t, c_t) - U_x(t, X_t^{\kappa, c})c_t)dt \\ \text{where } \mathcal{Q}(t, x, \kappa) &= \|x\kappa\|^2 + 2x\kappa \cdot \left(\frac{U_x(t, x)\eta_t^{\mathcal{R}} + \gamma_x(t, x)}{U_{xx}(t, x)} \right). \end{aligned}$$

Since $\kappa \in \mathcal{R}$, $\mathcal{Q}(t, x, \kappa)$ is only depending on $\gamma_x^{\mathcal{R}}(t, x)$, the orthogonal projection of $\gamma_x(t, x)$ on \mathcal{R}_t . The minimum $\mathcal{Q}^*(t, x) = \inf_{\kappa \in \mathcal{R}} \mathcal{Q}(t, x, \kappa)$ of the quadratic form $\mathcal{Q}(t, x, \kappa)$ is achieved at the minimizing policy $\bar{\kappa}$ given by

$$\begin{cases} x\bar{\kappa}_t(x) &= -\frac{1}{U_{xx}(t, x)}(U_x(t, x)\eta_t^{\mathcal{R}} + \gamma_x^{\mathcal{R}}(t, x)) \\ \mathcal{Q}^*(t, x) &= -\frac{1}{U_{xx}(t, x)^2}\|U_x(t, x)\eta_t^{\mathcal{R}} + \gamma_x^{\mathcal{R}}(t, x)\|^2 = -\|x\bar{\kappa}_t(x)\|^2. \end{cases} \quad (4.5)$$

(ii) By the Fenchel convexity inequality, the term in the third line is bounded by above by $V(t, c_t) - U_x(t, X_t^{\kappa, c})c_t \leq \tilde{V}(t, -U_x(t, X_t^{\kappa, c}))$.

For the second line, since $\mathcal{Q}(t, x, \kappa_t) \geq \mathcal{Q}^*(t, x) = -\|x\bar{\kappa}_t(x)\|^2$ and $U_{xx} \leq 0$, the term may be bounded by above by $(\beta(t, X_t^{\kappa, c}) + U_x(t, X_t^{\kappa, c})r_t X_t^{\kappa, c} - \frac{1}{2}U_{xx}(t, X_t^{\kappa, c})\|X_t^{\kappa, c}\bar{\kappa}_t(X_t^{\kappa, c})\|^2)$.

Then, if β satisfies the HJB constraint (4.4), the drift term is nonpositive for any $\kappa \in \mathcal{R}$ and $c \geq 0$, and the process $(U(t, X_t^{\kappa, c}) + \int_0^t V(s, c_s) ds)$ is a supermartingale.

(iii) Assume now that the wealth SDE associated with $(\bar{\kappa}, \bar{\zeta})$ admits a positive solution \bar{X} . Then, the non positive drift in the previous equation is equal to 0, so that $(U(t, \bar{X}_t) + \int_0^t V(s, \bar{\zeta}(s, \bar{X}_s)) ds)$ is a local martingale. This equality proves the existence of an optimal strategy $x\bar{\kappa}$ and $\bar{\zeta}$, and that \bar{X} is an optimal process. So, we have no reason to distinguish between processes with $-$ and processes with $*$. \square

Conjugate of consistent progressive utility with consumption. Let (\mathbf{U}, \mathbf{V}) be a pair of stochastic \mathcal{X}^c -consistent utilities with optimal strategy (κ^*, c^*) leading to the non negative wealth process $X^* = X^{\kappa^*, c^*}$. Convex analysis showed the interest to study the convex conjugate utilities $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$. Indeed, under mild regularity assumption, we have the following results (Karatzas-Shreve [17], Rogers [36]).

- (i) For any admissible state price density process $Y^\nu \in \mathcal{Y}$ with $\nu \in \mathcal{R}^\perp$, $(\tilde{U}(t, Y_t^\nu) + \int_0^t \tilde{V}(s, Y_s^\nu) ds)$ is a submartingale, and there exists a unique optimal process $Y^* := Y^{\nu^*}$ with $\nu^* \in \mathcal{R}^\perp$ such that $(\tilde{U}(t, Y_t^*) + \int_0^t \tilde{V}(s, Y_s^*) ds)$ is a local martingale. To summarize $\mathcal{U}(s, Y_s) = \text{ess sup}_{Y \in \mathcal{Y}} \mathbb{E} \left((\tilde{U}(t, Y_t^\nu) + \int_0^t \tilde{V}(s, Y_s^\nu) ds) / \mathcal{F}_s \right)$, a.s.
- (ii) *Optimal Processes characterization* Under regularity assumption, first order conditions imply some links between optimal processes, including their initial conditions,

$$Y_t^*(y) = U_x(t, X_t^*(x)) = V_c(t, c_t^*(c_0)), \quad y = u_x(x) = v_c(c_0). \quad (4.6)$$

The characteristics of the consistent conjugate progressive utility $\tilde{\mathbf{U}}$ with consumption can be also computed directly from Theorem 2.3, using a PDE approach. Given that the drift β is associated with an optimization program, it is easy to show that $\tilde{\beta}$ is also constrained by a HJB type relation in the new variables, and the convex conjugate utility system $(\tilde{\mathbf{U}}, \tilde{\mathbf{V}})$ is consistent (in a sense to be precised) with a family of state price density processes (Definition 4.2).

Theorem 4.5. *Let (\mathbf{U}, \mathbf{V}) a consistent progressive utility system with consumption, such that U is $\mathcal{K}_{loc}^{3, \delta}$ -regular ($\delta > 0$) with local characteristics (β, γ) satisfying Assumptions of Theorem 4.4. Then*

- (i) *The progressive convex conjugate utility $\tilde{\mathbf{U}}$ and its marginal conjugate utility $\tilde{\mathbf{U}}_{\mathbf{y}}$ are Itô random fields with local characteristics $(\tilde{\beta}, \tilde{\gamma})$ and $(\tilde{\beta}_y, \tilde{\gamma}_y)$ respectively.*
- (ii) *The local characteristics of the convex conjugate $\tilde{\mathbf{U}}$ are given by:*

$$\begin{cases} \tilde{\gamma}(t, y) := \gamma(t, -\tilde{U}_y(t, y)), & \tilde{\gamma}_y(t, y) := -\gamma_x(t, -\tilde{U}_y(t, y)) \cdot \tilde{U}_{yy}(y) \\ \tilde{\beta}(t, y) = y \tilde{U}_y(t, y) r_t + \frac{1}{2} \tilde{U}_{yy}(t, y) \|\tilde{\sigma}^*(t, y)\|^2 - \sigma_t^*(-\tilde{U}_y(t, y)) \cdot y \eta_t^{\mathcal{R}} - \tilde{V}(t, y) \end{cases} \quad (4.7)$$

- (iii) *For any admissible state price density process $Y^\nu \in \mathcal{Y}$ with $\nu \in \mathcal{R}^\perp$, $(\tilde{U}(t, Y_t^\nu) + \int_0^t \tilde{V}(s, Y_s^\nu) ds)$ is a submartingale, and a local martingale for any solution Y^* (if there ex-*

ists) of the equation $dY_t^* = Y_t^*[-r_t dt + (\nu^*(t, Y_t^*) - \eta_t^R).dW_t] = \tilde{\mu}^*(t, Y_t^*)dt - \tilde{\sigma}^*(t, Y_t^*).dW_t$, with $\tilde{\sigma}^*(t, y) = y(\nu_t^*(y) - \eta_t^R)$ and $\tilde{\mu}^*(t, y) = -r_t y$.

Proof. A similar proof in the framework without consumption can be found in [21]. \square

From now on, either the notation $\sigma_t^*(y)$ or $\sigma^*(t, y)$ will be used. To fix the idea, we now give the example of consistent Power Utilities, for which we prove the existence of optimal processes without any additional regularity conditions.

4.3 Consistent Power Utilities

Power utilities with constant risk aversion are widely used in economics, in particular for the Ramsey rule established in the next Section. It is also a useful example in the framework of forward utilities for its simplicity and its easy interpretation of the coefficient. To characterise such utilities, we start with a problem without consumption.

Consistent Progressive Power Utility without consumption see [20] for more details.

(i) Let us consider a consistent power utility $U^{(\alpha)}(t, x) = Z_t^{(\alpha)} \frac{x^{1-\alpha}}{1-\alpha}$ where $\alpha \in (0, 1)$ is the risk aversion coefficient and $Z^{(\alpha)}$ a semimartingale allowing to satisfy the consistency property. Then, the conjugate function $\tilde{U}^{(\alpha)}(t, y)$ satisfied $\tilde{U}^{(\alpha)}(t, y) = \tilde{Z}_t^{(\alpha)} \frac{y^{1-\frac{1}{\alpha}}}{\frac{1}{\alpha}-1}$. Since the risk aversion coefficient is given, we do not recall it if it not necessary.

(ii) Thanks to the consistency property, there exists an optimal wealth process $X^*(x)$ such that $U^{(\alpha)}(t, X_t^*(x)) = \frac{1}{1-\alpha} Z_t (X_t^*(x))^{1-\alpha}$ is a martingale, and such that $U_x^{(\alpha)}(t, X_t^*(x)) = Y_t^*(x^{-\alpha})$ is a state price density process with initial condition $x^{-\alpha}$.

(iii) In particular, using the intuitive factorization $Z_t = Z_t^R \cdot Z_t^\perp$ where Z_t^\perp is an exponential martingale $\mathcal{E}_t(\delta^\perp \cdot W)$ with $\delta^\perp \in \mathcal{R}^\perp$,

we see that $Z_t^R (X_t^*(x))^{-\alpha} = x^{-\alpha} Y_t^0$, where Y_t^0 is the minimal state price density.

The optimal wealth $X_t^*(x)$ and the optimal dual process $Y_t^*(y) = y Y_t^*$ are linear with respect to their initial condition. So,

$$(I) \left\{ \begin{array}{ll} Z_t = Z_t^\perp Y_t^0 (X_t^*)^\alpha, & \tilde{Z}_t = X_t^* (Y_t^*)^{\frac{1}{\alpha}} \\ X_t^*(x) = x X_t^*, & Y_t^*(y) = y Y_t^* = y Z_t^\perp Y_t^0 \\ U^{(\alpha)}(t, x) = \frac{Y_t^* X_t^*}{1-\alpha} \left(\frac{x}{X_t^*} \right)^{1-\alpha}, & \tilde{U}^{(\alpha)}(t, y) = \frac{Y_t^* X_t^*}{\frac{1}{\alpha}-1} \left(\frac{y}{Y_t^*} \right)^{1-\frac{1}{\alpha}} \end{array} \right.$$

Since the characteristics of the power utility $U^{(\alpha)}(t, x) = Z_t \frac{x^{1-\alpha}}{1-\alpha} = Z_t u^{(\alpha)}(x)$ are only dependent of the characteristics (β^Z, γ^Z) of Z ,

$$\gamma^{(\alpha)}(t, x) = \gamma_t^Z u^{(\alpha)}(x), \quad \beta^{(\alpha)}(t, x) = \beta_t^Z u^{(\alpha)}(x), \quad \text{with} \quad dZ_t = \beta_t^Z dt + \gamma_t^Z \cdot dW_t.$$

Equations (4.3) and (4.4) of Theorem 4.4 are easily verified (with $V = \tilde{V} = 0$), from the formula $Z_t = Y_t^* (X_t^*)^\alpha$ whose differential characteristics are $\gamma_t^Z = Z_t(\alpha \kappa_t^* + (\nu_t^* - \eta_t^R))$ and $\beta_t^Z = (1 - \alpha)Z_t(-r_t + \frac{1}{2}\alpha \|\kappa_t^*\|^2)$.

Consistent Progressive Power Utility with consumption When the problem consists in optimizing also a consumption process, we have to precise what stochastic utility for the consumption we must choose to satisfy the consistency of the utility system $(U^{(\alpha)}, V)$ when $U_t^{(\alpha)}(x) = \widehat{Z}_t u^{(\alpha)}(x)$ is a power progressive utility, and \widehat{Z} a semimartingale with local characteristics $(\hat{\gamma}, \hat{\beta})$. A useful tool is the system of equations (4.10), since the equation characterizing the process $\gamma_x(t, x) = \hat{\gamma}_t u_x^{(\alpha)}(x)$ does not depend explicitly on V . To make the distinction between the two problems, we introduce the symbol $\hat{\cdot}$ in the quantities relative to the problem with consumption.

(i) Equation (4.3), after dividing the both sides by $u_x^{(\alpha)}(x)$, yields to

$$\hat{\gamma}_t = \widehat{Z}_t (\alpha x^{-1} \hat{\sigma}_t^*(x) + (\hat{\nu}_t^*(U_x(t, x)) - \eta_t^{\mathcal{R}}).$$

Since $\hat{\gamma}_t$ does not depends on x , this equality implies as above that $x^{-1} \hat{\sigma}_t^*(x) = \hat{\kappa}_t^*(x)$ and $\hat{\nu}_t^*(U_x(t, x))$ does not depend on x , so as in the situation without consumption $\hat{\gamma}_t = \widehat{Z}_t (\alpha \hat{\kappa}_t^* + (\hat{\nu}_t^* - \eta_t^{\mathcal{R}}))$.

(ii) The drift equation (4.4) becomes

$$\hat{\beta}_t = \widehat{Z}_t (-(1 - \alpha) r_t + \frac{1}{2} \alpha (1 - \alpha) \|\hat{\kappa}_t^*\|^2) - \tilde{V}(t, \widehat{Z}_t u_x^{(\alpha)}(x)) / u^{(\alpha)}(x).$$

Thus, by the same argument as before, the progressive conjuguate utility $\tilde{V}(t, y)$ must be chosen in such a way that $\tilde{V}(t, \widehat{Z}_t u_x^{(\alpha)}(x)) = \alpha \hat{\psi}_t \widehat{Z}_t u^{(\alpha)}(x)$, where $\hat{\psi}_t$ is a positive adapted process with good integrability property. As a consequence, $\tilde{V}(t, y) = \hat{\psi}_t (\widehat{Z}_t)^{1/\alpha} \tilde{u}^{(\alpha)}(y)$. So, \tilde{V} is the Fenchel transform of a power utility $V(t, x) = (\hat{\psi}_t)^\alpha \widehat{Z}_t u^{(\alpha)}(x) = (\hat{\psi}_t)^\alpha U^{(\alpha)}(t, x)$.
(iii) Then, the process \widehat{Z} is a solution of the stochastic differential equation

$$d\widehat{Z}_t = \widehat{Z}_t [(\alpha \hat{\kappa}_t^* + (\hat{\nu}_t^* - \eta_t^{\mathcal{R}}) \cdot dW_t + (- (1 - \alpha) r_t + \frac{1}{2} \alpha (1 - \alpha) \|\hat{\kappa}_t^*\|^2 - \alpha \hat{\psi}_t) dt]$$

where the optimal strategies are the same as in the case without consumption ($\hat{\kappa}^* \equiv \kappa^*$, $\hat{\nu}^* \equiv \nu^*$), $\widehat{Z}_t = Z_t e^{-\alpha \int_0^t \hat{\psi}_s ds}$. The process $\hat{\psi}_t$ plays in this formula the role of an additional spread to the interest rate r_t for the wealth but not for the sate price density. This interpretation is justified by the closed form of the optimal consumption $c_t^*(x) = -\tilde{V}_y(t, \widehat{Z}_t u_x^{(\alpha)}(x))$ given after some tedious calculation by $c_t^*(z) = z \hat{\psi}_t$.

Corollary 4.6. *A consumption consistent progressive power utility system is necessarily a pair of power utilities with the same risk aversion coefficient α such that*

$$U^{(\alpha)}(t, x) = \widehat{Z}_t \frac{x^{1-\alpha}}{1-\alpha} = \widehat{Z}_t u^{(\alpha)}(x) \quad \text{and} \quad V^{(\alpha)}(t, x) = (\hat{\psi}_t)^\alpha U^{(\alpha)}(t, x).$$

(i) *The optimal processes are linear with respect of their initial condition, i.e.*

$$\widehat{X}_t^*(x) = x \widehat{X}_t^*, \quad Y_t^*(y) = y Y_t^*, \quad \text{and} \quad c_t^*(z) = z \hat{\psi}_t.$$

(ii) *The coefficient \widehat{Z}_t is determined by the optimal processes via $\widehat{Z}_t = Y_t^* (\widehat{X}_t^*)^\alpha$, while the coefficient $\hat{\psi}_t$ is only assumed to be positive.*

(iii) *The optimal processes (with initial condition 1) are driven by the system $c_t^* = \hat{\psi}_t$ and*

$$d\widehat{X}_t^* = \widehat{X}_t^* ((r_t - \hat{\psi}_t) dt + \kappa_t^* \cdot (dW_t + \eta_t^{\mathcal{R}})), \quad dY_t^* = Y_t^* (-r_t dt + (\nu_t^* - \eta_t^{\mathcal{R}}) dW_s) \quad (4.8)$$

In the general case of consistent progressive utilities, additional regularity conditions are needed, but it is still possible to give a closed form of the forward utility in terms of initial condition and optimal processes.

4.4 Regularity issues for existence of consistent progressive utility and closed form characterization via optimal processes.

In Subsection 4.2, we have assumed the consistent progressive utility \mathbf{U} sufficiently regular to apply Itô's Ventzel formula in view of establishing HJB constraint; then we have shown the links between local characteristics and coefficients of the SDE associated with an optimal portfolio, without proving the existence. The same kind of assumptions are made on the conjugate $\tilde{\mathbf{U}}$, implying the dual HJB constraint in the same way than for the primal problem. But it is well-known that in all generality these assumptions are not satisfied. Assuming the existence of regular progressive utility satisfying HJB constraint, we show that (\mathbf{U}, \mathbf{V}) is a \mathcal{X}^c -consistent stochastic utility system, associated with a regular optimal dual SDE $(\tilde{\mu}^*, \tilde{\sigma}^*)$ whose coefficients are based only on the diffusion characteristics γ of \mathbf{U} , and do not depend on the utility of consumption process V . The existence of this strong dual solution is very important in view to apply Theorem 3.5 not directly to \mathbf{U} but to \mathbf{U}_x whose the diffusion characteristic γ_x has the same form than the diffusion characteristic of the random field G , where σ^Y is replaced by $\tilde{\sigma}_t^*$, and σ^X by $\sigma_t^*(x) = \kappa_t^*(x)$. In addition to consistency, under this HJB constraint, we show that such utility system can be represented in a closed form.

To be closed to the notation of Theorem 3.5, we recall all the coefficients of optimal SDEs associated with the primal and dual problems,

$$\begin{cases} \tilde{\sigma}_t^*(y) &:= y(\nu_t^*(y) - \eta_t^R), & \tilde{\mu}_t^*(y) &:= -r_t y, \\ \sigma_t^*(x) &:= x\kappa_t^*(x) & \mu_t^{*,c}(x) &:= r_t x + x\kappa_t^*(x) \cdot \eta_t^R - \zeta^*(t, x), \\ \hat{L}_{t,x}^{(\mu^*, \sigma^*, c)} &:= \frac{1}{2} \partial_x (\|\sigma_t^*(x)\|^2 \partial_x) - \mu_t^{*,c}(x) \partial_x, & L_{t,x}^{(\tilde{\mu}^*, \tilde{\sigma}^*)} &= \frac{1}{2} \|\sigma_t^*(x)\|^2 \partial_{xx} + \tilde{\mu}_t^*(x) \partial_x \end{cases} \quad (4.9)$$

Proposition 4.7. *Let \mathbf{U} be a $\mathcal{K}_{loc}^{2,\delta}$ -regular ($\delta > 0$) progressive utility \mathbf{U} , whose local characteristics (β, γ) satisfy the HJB constraints,*

$$\begin{cases} \gamma_x(t, x) &:= -U_{xx}(t, x)x\kappa_t^*(x) + \tilde{\sigma}_t^*(U_x(t, x)), & \gamma_x^\perp(t, -\tilde{U}_y(t, y)) &= y\nu_t^*(y) \\ \beta(t, x) &:= -U_x(t, x)xr_t - \tilde{V}(t, U_x(t, x)) + \frac{1}{2}U_{xx}(t, x)\|x\kappa_t^*(x)\|^2 \end{cases} \quad (4.10)$$

(i) *The marginal utility \mathbf{U}_x is a decreasing solution of the SPDE(3.4) with coefficients $(\mu^{*,c}, \sigma^*)$ and $(\tilde{\mu}^*, \tilde{\sigma}^*)$*

$$\begin{aligned} dU_x(t, x) &= \tilde{\sigma}_t^*(U_x(t, x)) \cdot dW_t + \tilde{\mu}_t^*(U_x(t, x)) dt \\ &\quad - \partial_x U_x(t, x) \sigma_t^*(x) \cdot (dW_t + \tilde{\sigma}_y^*(t, U_x(t, x)) dt) + \hat{L}_{t,x}^{*,c}(U) dt \end{aligned} \quad (4.11)$$

(ii) *Assume that the SDE $(\mu^{*,c}, \sigma_t^*)$ and SDE $(-r_t y, \tilde{\sigma}^*(t, y))$ admit a monotonic solution*

$(X_t^*(x), Y_t^*(y))$. Then, the marginal forward utility at time t is the non linear transportation of the marginal utility at time 0 through the optimal dual processes,

$$U_x(t, x) = Y_t^*(u_x((X_t^*)^{-1}(x))) \quad (4.12)$$

Proof. First, as \mathbf{U} is assumed to be $\mathcal{K}_{loc}^{2,\delta} \cap \mathcal{C}^3$ -regular, \mathbf{U}_x is of class $\mathcal{K}_{loc}^{1,\delta}$ and its local characteristics (β_x, γ_x) are of class \mathcal{C}^1 in x ; then, the vectors $\sigma_t^*(x) = -(\gamma_x^\mathcal{R}(t, x) + \eta_t^\mathcal{R} U_x(t, x))/U_{xx}(t, x)$ and $\tilde{\sigma}_t^*(y) = \gamma_x^\perp(t, -\tilde{U}_y(t, y)) - y\eta_t^\mathcal{R}$ are also of class \mathcal{C}^1 , necessary condition to define $\hat{L}^{*,c}$.

By derivation of the local characteristics of the regular progressive utility \mathbf{U} , we see that

$$\beta_x(t, x) = -\partial_x(U_x(t, x)xr_t) - U_{xx}(t, x)\tilde{V}_y(t, U_x(t, x)) + \partial_x\left(\frac{1}{2}U_{xx}(t, x)\|\sigma_t^*(x)\|^2\right).$$

Observing that $\tilde{V}_y(t, U_x(t, x)) = -(V_c)^{-1}(t, U_x(t, x)) = -\zeta^*(t, x)$, it follows that

$$\begin{aligned} -\partial_x(U_x(t, x)xr_t) - U_{xx}(t, x)\tilde{V}_y(t, U_x(t, x)) &= -\partial_x(U_x(t, x)xr_t + U_{xx}(t, x)\zeta^*(t, x)) \\ &= -\partial_x U_x(t, x)\mu^{*,c}(t, x) - r_t U_x(t, x). \end{aligned}$$

It remains to make some slight transformations on the drift characteristic:

$$\begin{aligned} \beta_x(t, x) &= -\partial_x U_x(t, x)\mu^{*,c}(t, x) - r_t U_x(t, x) + \partial_x\left(\frac{1}{2}U_{xx}(t, x)\|\sigma_t^*(x)\|^2\right) \\ &= \hat{L}_{t,x}^{*,c}(U) - r_t U_x(t, x) + \partial_x U_x(t, x)\sigma_t^*(x).\eta_t^\mathcal{R} \\ &= \hat{L}_{t,x}^{*,c}(U) + \tilde{\mu}_t^*(U_x(t, x)) + \partial_x U_x(t, x)\sigma_t^*(x).\eta_t^\mathcal{R} \end{aligned}$$

Let us give another interpretation of $\sigma_t^*(x).\eta_t^\mathcal{R}$. Since $\tilde{\sigma}^*(t, y) + \eta_t^\mathcal{R}y$ belongs to the vector space \mathcal{R}_t^\perp the spatial derivative $\tilde{\sigma}_y^*(t, y) + \eta_t^\mathcal{R}$ is also in \mathcal{R}_t^\perp , yielding to the relation on the scalar products $-\sigma_t^*(x).\eta_t^\mathcal{R} = \sigma_t^*(x).\tilde{\sigma}_y^*(t, y)$. Then, Identity (4.11) holds true.

(ii) If we know the existence of monotonic solution of $\text{SDE}(\mu^{*,c}, \sigma_t^*)$ and $\text{SDE}(-r_t y, \tilde{\sigma}^*(t, y))$, from the form of the SPDE associated with U_x and the assertion (ii) of Theorem 3.5, we easily obtained the representation $U_x(t, x) = Y_t^*(u_x((X_t^*)^{-1}(x)))$. \square

The next theorem gives sufficient condition for the existence of (monotonic) optimal solutions for the optimisation problem.

Theorem 4.8. Let \mathbf{U} be a $\mathcal{K}_{loc}^{2,\delta} \cap \mathcal{C}^3$ -regular ($\delta > 0$) progressive utility \mathbf{U} , whose local characteristics (β, γ) satisfy HJB constraints 4.10 .

MAIN RESULT Suppose the existence of two adapted bounds $(K^1, K^2) \in \mathcal{L}^2(dt)$ such that the regular random field γ_x^\perp satisfy

$$\|\gamma_x^\perp(t, x)\| \leq K_t^1 |U_x(t, x)|, \|\gamma_{xx}^\perp(t, x)\| \leq K_t^2 |U_{xx}(t, x)|, \text{ a.s.}, \quad (4.13)$$

(i) As $y\nu_t^*(y) = \gamma_x^\perp(t, U_x^{-1}(t, y))$, and $\tilde{\sigma}_t^*(y) := y(\nu_t^*(y) - \eta_t^\mathcal{R})$, the $\text{SDE}(-r_t y, \tilde{\sigma}^*(t, y))$ is uniformly Lipschitz and its unique strong solution $Y_t^*(y)$ is increasing, with range $[0, \infty)$.

(ii) Moreover, assume the existence of an adapted bound K^3 such that process $V_c(t, K^3 x) \geq U_x(t, x)$ a.s. for any x . Using the notations $\sigma_t^*(x) := x\kappa_t^*(x)$ and $\mu_t^{*,c}(x) := r_t x + x\kappa_t^*(x).\eta_t^\mathcal{R} - \zeta^*(t, x)$,

a) The $\text{SDE}(\mu^{*,c}, \sigma^*)$ is locally Lipschitz and admits a maximal positive monotonic solution X^* such that $U_x(\cdot, X^*(x))$ is distinguishable from the solution $Y_\cdot^*(u_x(x))$.

b) The optimal consumption along the optimal wealth process is

$$c_t^*(x) = \zeta^*(t, X_t^*(x)) = -\tilde{V}_y(t, U_x(t, X_t^*(x))) = -\tilde{V}_y(t, Y_t^*(u_x(x))).$$

REVERSE SOLUTION Denote by $\mu^*(t, x) = r_t x + x\kappa_t^*(x)\eta_t^R$ the drift of some portfolio without consumption, by $\bar{\zeta}(t, x)$ some increasing adapted positive random field, and by $\bar{\mu}(t, x) = \mu^*(t, x) - \bar{\zeta}(t, x)$. Assume the existence (\bar{X}, Y^*) of two monotonic solutions of $SDE(\bar{\mu}, \sigma^*)$ and $SDE(\bar{\mu}^*, \bar{\sigma}^*)$ with range $(0, \infty)$.

a) For any deterministic utility function (u, v) such that $v_c(\zeta(0, x)) = u_x(x)$,

$$U_x(t, x) = Y_t^*(u_x(\bar{X}_t^{-1}(x))), \quad V_c(t, c) = U_x(t, \zeta^{*, -1}(t, c))$$

b) Moreover, if $Y_t^*(u(x))\partial_x \bar{X}_t(x)$ is Lebesgue-integrable in a neighborhood of 0, then

$$U(t, \bar{X}_t(x)) = \int_0^x Y_t^*(u_x(z))\partial_x \bar{X}_t(z)dz, \quad V(t, \zeta^*(t, c)) = \int_0^c U_x(t, z)dz \zeta^*(t, z) \quad (4.14)$$

Then, with these additional integrability assumptions, $(\mathbf{U}_x, \mathbf{V}_c)$ are the marginal utilities of a consistent utility system with consumption.

Proof. The proof of the MAIN RESULT is easy, given the previous results.

(i) This assertion is a simple consequence of assumptions on the orthogonal diffusion characteristics.

(ii) a) We start by solving the wealth SDE with coefficients $\sigma_t^*(x)$ and $\mu_t^{*,c}(x)$. These coefficients are locally Lipschitz, with linear growth since $\zeta^*(t, x) = (V_c)^{-1}(t, U_x(t, x)) \leq K_t^3 x$. Then a strong solution X^* exists up to a explosion time $\tau(x)$. But, by verification from the SPDE, $U_x(t, X_t^*(x)) = Y_t^*(u_x)$ on $[0, \tau(x))$. Since $Y_t^*(u_x)$ is well defined, $\tau(x) = +\infty$ a.s..

The formulation of the REVERSE PROBLEM WITH CONSUMPTION is a more complex, since we have to take into account the increasing function ζ . The assertion a) is proved by the same argument as before using Theorem 3.5. The assertion b) gives an intuitive form to the construction of the forward utility itself by application of the change of variable formula. \square

4.5 Value function of backward classical utility maximization problem as consistent progressive utility

This subsection points out the similarities and the differences between consistent progressive utilities and backward classical value functions, and their corresponding portfolio/consumption optimization problems.

Classical portfolio/consumption optimization problem and its conjugate problem The classic problem of optimizing consumption and terminal wealth is determined by a fixed horizon T_H and two deterministic utility functions $u(\cdot)$ and $v(t, \cdot)$ defined up to this horizon. Using the same notations as in Section 4, the classical optimization problem is formulated as the following maximization problem,

$$\sup_{(\kappa, c) \in \mathcal{X}^c} \mathbb{E} \left(u(X_{T_H}^{\kappa, c}) + \int_0^{T_H} v(t, c_t) dt \right). \quad (4.15)$$

For any $[0, T_H]$ -valued \mathbb{F} -stopping τ and for any positive random variable \mathcal{F}_τ -mesurable ξ_τ , $\mathcal{X}^c(\tau, \xi_\tau)$ denotes the set of admissible strategies starting at time τ with an initial positive wealth ξ_τ , stopped when the wealth process reaches 0. The corresponding value system (that is a family of random variables indexed by (τ, ξ_τ)) is defined as,

$$\mathcal{U}(\tau, \xi_\tau) = \text{ess sup}_{(\kappa, c) \in \mathcal{X}^c(\tau, \xi_\tau)} \mathbb{E} \left(u(X_{T_H}^{\kappa, c}(\tau, \xi_\tau)) + \int_\tau^{T_H} v(s, c_s) ds | \mathcal{F}_\tau \right), \text{ a.s.} \quad (4.16)$$

with terminal condition $\mathcal{U}(T_H, x) = u(x)$.

We assume the existence of a progressive utility (still denoted $\mathcal{U}(t, x)$) aggregating these system: this result is more or less implicit in the literature and has been proven by Englezos and Karatzas [30] in the case of a complete market. The proof for an incomplete market will be done in a future work.

As it is classical in such stochastic control problems ([18]) and shown by W. Schachermayer in [37] for problem without consumption, the dynamic programming principle reads as follows: for any pair $\tau \leq \vartheta$ of $[0, T_H]$ -valued stopping times

$$\mathcal{U}(\tau, \xi_\tau) = \text{ess sup}_{(X, c) \in \mathcal{X}^c(\tau, \xi_\tau)} \mathbb{E} \left(\mathcal{U}(\vartheta, X_\vartheta(\tau, \xi_\tau) + \int_\tau^\vartheta v(s, c_s) ds | \mathcal{F}_\tau \right) \text{ a.s.}$$

Under mild assumptions on the asymptotic elasticity of utility functions (u, v) , it is also proved in [23] and [37] the existence for any initial wealth of an optimal solution (portfolio, consumption). Then, $(\mathcal{U}(t, x), v(t, c))$ is a \mathcal{X}^c -consistent dynamic utility system in the sense of Definition 4.3 up to time T_H . The same property was proved by Mania and Tevzadze [26] in a problem without consumption under strong regularity assumption on the value function \mathcal{U} by using backward SDPE.

Similarly, let $(\tilde{\mathcal{U}}(t, y), \tilde{v}(t, y))$ be the convex conjugate utilities of $(\mathcal{U}(t, x), v(t, c))$. $\tilde{\mathcal{U}}(t, y)$ aggregates the dynamic version of the equivalent backward dual problem (Karatzas-Lehoczky-Shreve [16]) defined, for any \mathbb{F} -stopping time $\tau \leq T_H$ and for any positive random variable \mathcal{F}_τ -mesurable ψ_τ , from the family $\mathcal{Y}^c(\tau, \psi_\tau)$ of the state price density processes $\{Y^\nu, \nu \in \mathcal{R}^\perp\}$ (see (4.2)) with dynamics $dY_t^\nu(y) = Y_t^\nu(y)[-r_t dt + (\nu_t - \eta_t^\mathcal{R}).dW_t]$, $\nu_t \in \mathcal{R}_t^\perp$, starting from ψ_τ at time τ . The value function of the dual backward optimization problem is then

$$\tilde{\mathcal{U}}(\tau, \psi_\tau) = \text{ess inf}_{Y^\nu \in \mathcal{Y}^c(\tau, \psi_\tau)} \mathbb{E} \left(\tilde{u}(Y_{T_H}^\nu) + \int_\tau^{T_H} \tilde{v}(s, Y_s^\nu) ds | \mathcal{F}_\tau \right), \text{ a.s.} \quad (4.17)$$

Optimal processes The process $(Y_t^0(y))$ associated with $\nu = 0$ is linear in y , and denoted $Y_t^0(y) = yY_t^0(1) = yY_t^0$, for simplicity. We also frequently used the shorthand notation $Y_{s,t}^0 = Y_t^0/Y_s^0$, $s < t$.

(i) In a complete market, (yY_t^0) is the unique state price density process, and the value function is given by

$$\tilde{\mathcal{U}}(\tau, \psi_\tau) = \mathbb{E} \left(\tilde{u}(\psi_\tau Y_{\tau, T_H}^0) + \int_\tau^{T_H} \tilde{v}(s, \psi_\tau Y_{\tau, s}^0) ds | \mathcal{F}_\tau \right), \text{ P.a.s.}$$

Then the optimal state price density Y^* does not depend on the utility functions u et v , and on the horizon T_H , to the difference of the optimal processes (X^*, c^*) .

(ii) In an incomplete market, we refer to [23] to ensure the existence of an optimal state price density Y^{ν^*, T_H} minimizing the criterium (4.17) (observe that this can be done under weaker assumptions than in the forward case). Now, the optimal choice depends on the horizon and obviously on the utility functions (u, v) . Since we are essentially interested in the horizon dependency, we do not recall the influence of the utility criterium. To avoid too cumbersome formula, Y^{ν^*, T_H} is often denoted $Y^{*, H}$.

Since (\tilde{U}, \tilde{v}) is a progressive conjugate utility system, the results of Theorem 4.7, (obtained directly in the aforementioned works by the maximum principle), we known that $U_x(t, X_t^{*, H}(x))$, $Y_t^{*, H}(y)$ and the optimal consumption rate $c_t^{*, H}(c_0)$ are linked by their initial conditions: $\mathcal{U}_x(0, x) = y = v_c(0, c_0)$

$$\begin{cases} c_t^{*, H}(c_0) &= -\tilde{v}_y(t, Y_t^{*, H}(y)) \quad \text{i.e.} & v_c(t, c_t^{*, H}(y)) = Y_t^{*, H}(y), \\ \mathcal{U}_x(t, X_t^{*, H}(x)) &= Y_t^{*, H}(\mathcal{U}_x(0, x)) = v_c(t, c_t^{*, H}(-\tilde{v}_y(0, y))) & \mathcal{U}_x(0, x) = y = v_c(0, c_0) \end{cases} \quad (4.18)$$

But the way the classical optimization problem is posed is completely different from the progressive utility problem: the initial value of \mathcal{U} is computed through a backward analysis, starting from its given terminal value, whereas for progressive utilities the initial value is given. Moreover, the optimal wealth is characterized only from its terminal value $X_{T_H}^{*, H}$. To compute its value at any time t , we have to use pricing techniques based on the fact that $(X_{T_H}^* + \int_0^{T_H} c_s^* ds)$ is a replicable asset, and its market value at time $t \leq T_H$ is given by

$$X_t^{*, H}(x) = \mathbb{E}\left(Y_{T_H}^{*, H}(y)X_{T_H}^{*, H} + \int_t^{T_H} Y_s^{*, H}(y)c_s^{*, H}(c_0)ds \middle| \mathcal{F}_t\right). \quad (4.19)$$

An other major difference is that in the backward point of view no attention is paid to the monotony of optimal strategies.

Example of horizon dependency In contrast to the progressive utility framework, the optimal solution in the classical setting highly depends on the horizon T_H , which leads to intertemporality issues. To illustrate this time-inconsistency, let us consider an intermediate horizon T between 0 and T_H and the following two scenarios.

- In the first one, the investor computes his optimal strategy for the horizon T and the utility functions (u, v) , and then reinvests at time T its wealth X_T^* to realize an optimal strategic policy $(X_{T_H}^{*, H}(T, X_T^*), c_t^{*, H}(T, X_T^*))$, optimal between the dates (T, T_H) for the problem with utility functions (u^H, v) .
- In the second one, the investor computes his optimal strategy, denoted $(\hat{X}_{T_H}, \hat{c}_t^H)$, directly for the horizon T_H and the utility functions (u^H, v) .

By uniqueness of preferences, often implicitly assumed by the investors, the terminal value of both scenarios must coincide, that is $X_{T_H}^*(T, X_T^*) = \hat{X}_{T_H}$ a.s.. for any T and T_H ($T < T_H$). This is impossible in general. Indeed, between (T, T_H) the investor is using the same utility functions, (u^H, v) applied to different initial wealths at time T , X_T^*

for the first strategy, and \hat{X}_T^* for the second strategy, since $\hat{X}_{T_H}(T, \hat{X}_T) = \hat{X}_{T_H}$ *a.s.* In particular, if \hat{X} is monotonic with respect to the initial wealth, the final time consistency can be done if and only if $\hat{X}_T = X_T^*$, $\mathbb{P} - a.s.$. If we are looking for the same property at any time T , the wealth process \hat{X} and X^* are the same. On the other hand, the dynamic programming principle implies that \hat{X}_T is the optimal wealth for the classical problem with horizon T , but stochastic utility $(\mathcal{U}_x(T, x), v)$. In any case, the optimal strategies can not be the same.

Therefore, progressive utilities processes are an alternative to classical utilities functions that gives time-consistency properties, and motivate to reconsider problems issued from classical utility framework, with the light of intertemporal consistency. Section 5 focus on the example of long term discount rates and yield curves. But before this, as an application of utility maximization, we recall some results on the pricing of contingent claim in finance.

4.6 Risk neutral pricing and marginal utility (with consumption) indifference pricing

In the backward point of view, we have found the market value of the optimal wealth, by the so-called pricing rule (4.19). This question is related to a more general issue in finance, that consists in the pricing of a bounded contingent claim ζ_T , paid at date T , $T \leq T_H$.

Risk neutral pricing of hedgeable payoffs (i) In the study of optimal state price density in 4.5, we have seen the "universal" rule played by the so-called minimal density process $Y_t^0(y) = yY_t^0$. In particular, since \mathcal{R}_t is a vector space, money market strategies ($\kappa \equiv 0$) are admissible, and $L_t^0 = e^{\int_0^t r_s ds} Y_t^0$ is a local martingale. We now assume that (L_t^0) is a uniformly integrable martingale on $[0, T_H]$, which allows us to introduce a minimal, also called risk-neutral, martingale measure,

$$d\mathbb{Q} = L_{T_H}^0 \cdot d\mathbb{P} \text{ on the } \sigma\text{-field } \mathcal{F}_{T_H}.$$

More generally, for any admissible $\nu \in \mathcal{R}_t^\perp$, $L_t^\nu(y) = e^{\int_0^t r_s ds} Y_t^\nu(y) := L_t^0 L_t^{\perp, \nu}(y)$ is also a local martingale, product of the martingale L^0 and the orthogonal local martingale $L^{\perp, \nu}(y)$. So, $L^{\perp, \nu}(y)$ is a \mathbb{Q} -local martingale, with \mathbb{Q} -expectation smaller than y . When $\mathbb{E}(L_{T_H}^\nu(y)) = y$, then $L_{T_H}^\nu(y)/y$ is the density of a probability measure \mathbb{Q}^ν with respect to \mathbb{P} , and $L_{T_H}^{\perp, \nu}(y)/y$ is the density of \mathbb{Q}^ν with respect to \mathbb{Q} .

(ii) In complete market, or more generally in incomplete market without arbitrage opportunity, the market price $p^m(\zeta_T)$ (p^m when it is not ambiguous) of any bounded contingent claim ζ_T paid at date T that is replicable by an admissible self-financing portfolio is a bounded process p_t^m such that $Y_t^\nu p_t^m$ is a local martingale for any admissible state price density, in particular for yY_t^0 and $Y_t^*(y)$. Since L^0 is a true martingale, and ζ_T is bounded, $(Y_t^0 p_t^m)$ is also a true martingale given by the conditional expectation of its terminal value;

this observation yields to the classical pricing formula (in a complete market) as the minimal risk neutral conditional expectation of the discounted claim between t and T ,

$$p_t^m = \mathbb{E}\left[\frac{Y_T^0(y)}{Y_t^0(y)}\zeta_T|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\zeta_T|\mathcal{F}_t\right]. \quad (4.20)$$

Moreover since for any admissible process $\nu \in \mathcal{R}^\perp$, $(L_t^{\perp,\nu}(y)e^{-\int_t^T r_s ds}p_t^m)$ is also a positive \mathbb{Q} -local martingale, and then a \mathbb{Q} -supermartingale, the following inequality (with equality if $L^{\perp,\nu}$ is a \mathbb{Q} -martingale) holds true

$$\mathbb{E}\left[\frac{Y_T^\nu(y)}{Y_t^\nu(y)}\zeta_T|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{L_T^{\perp,\nu}(y)}{L_t^{\perp,\nu}(y)}e^{-\int_t^T r_s ds}\zeta_T|\mathcal{F}_t\right] \leq p_t^m, \quad \mathbb{P} - a.s. \quad (4.21)$$

The same pricing formula may be used for pricing bounded hedgeable pay-off. *The minimal risk-neutral pricing rule gives the maximal seller price for bounded hedgeable contingent claim.*

(iii) In the forward point of view, we know, from the regularity assumption, that the optimal state price Y^* admits the following decomposition $Y_t^*(y) = yY_t^0L_t^{\perp,*}(y)$, where $L_t^{\perp,*}(y)$ is a \mathbb{Q} -uniformly integrable martingale. Then, all the previous inequalities are equalities and in particular, for hedgeable payoff ζ_T ,

$$\mathbb{E}\left[\frac{Y_T^0(y)}{Y_t^0(y)}\zeta_T|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\zeta_T|\mathcal{F}_t\right] = p_t^m. \quad \mathbb{P} - a.s.$$

The same property holds true in the backward case, on the assumption that $L_t^{\perp,*,H}(y)$ is a \mathbb{Q} -uniformly integrable martingale.

Marginal utility indifference pricing When the payoff ζ_{T_H} is not replicable in incomplete market, there are different ways to evaluate the risk coming from the unhedgeable part, yielding to a bid-ask spread. A way is the pricing by indifference.

When the investors are aware of their sensitivity to the unhedgeable risk, they can try to transact for only a little amount in the risky contract. In this case, the buyer wants to transact at the buyer's "fair price" (also called *Davis price* or *marginal utility price* [4]), which corresponds to the zero marginal rate of substitution p_t^u . In other words, considering the two following backward maximization problems (with and without the claim ζ_{T_H}):

$$\mathcal{U}^\zeta(t, x, q) := \sup_{(\kappa, c) \in \mathcal{X}^c(t, x)} \mathbb{E}[U(T_H, X_{T_H}^\kappa + q\zeta_{T_H}) + \int_t^{T_H} V(s, c_s)ds|\mathcal{F}_t], \quad (4.22)$$

$$\mathcal{U}(t, x) := \sup_{(\kappa, c) \in \mathcal{X}^c(t, x)} \mathbb{E}[U(T_H, X_{T_H}^\kappa) + \int_t^{T_H} V(s, c_s)ds|\mathcal{F}_t], \quad t \leq T_H \quad (4.23)$$

the marginal utility *indifference price* is the price at which the investor is indifferent from investing or not in the contingent claim: it is the \mathcal{F}_t -adapted process $(p_t^u(x))_{t \in [0, T_H]}$ determined at any time t by the non linear relationship

$$\partial_q \mathcal{U}^\zeta(t, x, q)|_{q=0} = \partial_q \mathcal{U}(t, x + qp_t^u(x))|_{q=0}, \quad \text{for all } t \in [0, T_H]. \quad (4.24)$$

The marginal utility price is a linear pricing rule. Using this pricing rule means that there exists a consensus on this price for a small amount, but investors are not sure to

have liquidity at this price. In the backward case, the marginal utility price, such as the optimal state price density, depends on the horizon T_H . In particular, if the contingent claim ζ_T is delivered at time $T < T_H$, then ζ_T can be invested between time T and T_H into any admissible portfolio $X(T, \zeta_T)$ (martingale under Y^*) and computing the marginal utility price with terminal payoff $\zeta_{T_H} = X_{T_H}(T, \zeta_T)$ leads to the same price, as explained in the following proposition. When needed, we use the notation $Y^{*,H}$ and $p_0^{u,H}(x, \zeta_T)$ to emphasize the time horizon dependency of the backward optimization problem.

Proposition 4.9. *Let (U, V) be the progressive utilities associated with a consumption consistent optimization problem with optimal state price density process $Y^*(y)$.*

(i) *For any non negative contingent claim ζ_{T_H} delivered at time T_H , the marginal utility price (also called Davis-price) is given via the dual parametrization y*

$$p_t^u(x, \zeta_{T_H}) = \mathbb{E}[\zeta_{T_H} Y_{T_H}^*(t, y)/y | \mathcal{F}_t], \quad y = \mathcal{U}_x(t, x). \quad (4.25)$$

(ii) *In the forward case, the pricing rule may be defined for any maturity $T \leq T_H$ in the same way. Then, the pricing rule is time-consistent,*

$$p_t^u(x, \zeta_{T_H}) = p_t^u(x, \zeta_T(t, x)) \quad \text{where } \zeta_T(t, x) = p_T^u(X_T^*(t, x), \zeta_{T_H}).$$

(iii) *In the backward case, the marginal utility indifference price is only defined for cash-flow paid at horizon T_H . When the claim ζ_T is delivered at time T before T_H , ζ_T may be considered as the (indifference) price at T of any admissible portfolio starting from ζ_T at T with terminal wealth $X_{T_H}(T, \zeta_T) = \zeta_{T_H}$. The marginal utility price of ζ_T , denoted $p_0^{u,H}(x, \zeta_T)$ to recall its dependency in T_H is then,*

$$p_t^{u,H}(x, \zeta_T) = p_t^{u,H}(x, \zeta_{T_H}) = \mathbb{E}[\zeta_{T_H} Y_{T_H}^{*,H}(t, y)/y | \mathcal{F}_t] = \mathbb{E}[\zeta_T Y_T^{*,H}(t, y)/y | \mathcal{F}_t]. \quad (4.26)$$

(iv) *The backward marginal utility pricing is a well-posed pricing rule, since it is not depending on the choice of the admissible extension on ζ_T . Moreover, the rule is also time-consistent.*

Proof. Following Davis [4], we compute the marginal indifference price of any contingent claim as follows. Denote by $(X^{*,q}(z), c^{*,q}(z))$ the optimal strategy of the optimization program (4.22) (q quantity of claim ζ_{T_H}), i.e.

$$\mathbb{E}[U(T_H, X_{T_H}^{*,q}(x) + q\zeta_{T_H}) + \int_t^{T_H} V(s, c_s^{*,q}(x))ds] = \mathcal{U}^\zeta(t, x, q).$$

Formally, we can derive with respect to q under the expectation, and take the value of the derivative at $q = 0$ (known as the envelope theorem in economics)

$$\begin{aligned} \partial_q \mathcal{U}^\zeta(0, x, q)|_{q=0} &= \mathbb{E}\left[(U_x(T_H, X_{T_H}^{*,q}(x))(\partial_q X_{T_H}^{*,q}(x) + \zeta_{T_H})|_{q=0} \right. \\ &\quad \left. + \int_0^{T_H} (V_c(s, c_s^{*,q}(x))\partial_q c_s^{*,q}(x)|_{q=0})ds \right]. \end{aligned} \quad (4.27)$$

Under regularity assumption, it is shown in [4] that the optimal processes $(X_{T_H}^{*,q}, c_s^{*,q}(x))$ are continuously differentiable with respect to the quantity q satisfying $\lim_{q \rightarrow 0} X_{T_H}^{*,q} = X_{T_H}^*$,

and $\lim_{q \rightarrow 0} \partial_q X_{T_H}^{*,q} = 0, a.s.$; $\lim_{q \rightarrow 0} c_s^{*,q}(x) = c_s^*(x)$ and $\lim_{q \rightarrow 0} \partial_q c_s^{*,q}(x) = 0, a.s..$

This implies that the marginal indifference price satisfies

$$\mathbb{E}[\zeta_{T_H} U_x(T_H, X_{T_H}^*(x))] = p_0^u(x) \mathcal{U}_x(0, x).$$

In the forward and backward case, the marginal utility of the optimal wealth at the horizon T_H , $U_x(T_H, X_{T_H}^*(x))$, is the optimal state price density $Y_{T_H}^*(y)$ with initial condition $y = \mathcal{U}_x(0, x)$.

The main difference is that in the forward case, the process Y^* does not depend of T_H in contrast to the backward setting. In the forward case,

$$p_0^u(x) = \frac{1}{\mathcal{U}_x(0, x)} \mathbb{E}[U_x(T_H, X_{T_H}^*(x)) \zeta_{T_H}] = \mathbb{E}[\zeta_{T_H} Y_{T_H}^*(y)/y]. \quad (4.28)$$

In the backward case, if the maturity of the claim is $T \leq T_H$, then investing the amount ζ_T in any admissible portfolio $X(T, \zeta_T)$ such that $(X_t(T, \zeta_T) Y_t^{*,H}(y))_{t \geq T}$ is a martingale and taking $\zeta_{T_H} = X_{T_H}(T, \zeta_T)$, it follows that in any case

$$\begin{aligned} p_0^{u,H}(x, \zeta_T) &= p_0^{u,H}(x, \zeta_{T_H}) = \mathbb{E}[\mathbb{E}(X_{T_H}(T, \zeta_T) Y_{T_H}^{*,H}(y)/y | \mathcal{F}_T)] \\ &= \mathbb{E}[\zeta_T Y_T^{*,H}(y)/y], \quad y = \mathcal{U}_x(0, x) \end{aligned} \quad (4.29)$$

which proves that the backward marginal utility pricing is a well-posed pricing rule. \square

The same argument may be used at any date t to define the marginal utility price, using the conditional distribution with respect to the filtration \mathcal{F}_t ,

$$\begin{aligned} p_t^u(z) &= \frac{1}{\mathcal{U}_x(t, z)} \mathbb{E}[U_x(T_H, X_{T_H}^*(z)) \zeta_{T_H} | \mathcal{F}_t], \quad z = X_t^*(x) \\ &= \mathbb{E}[\zeta_{T_H} Y_{T_H}^*(\phi_t)/\phi_t | \mathcal{F}_t], \quad \phi = \mathcal{U}_x(t, X_t^*(x)) = Y_t^*(\mathcal{U}_x(0, x)). \end{aligned}$$

5 Application to yield curves dynamics

For financing of ecological projects reducing global warming, for longevity issues or any other investment with a long term impact, it is necessary to model accurately long run interest rates. The answer cannot find in financial market, since for longer maturities (30 years and more), the bond market becomes highly illiquid and standard financial interest rates models cannot be easily extended.

5.1 General macroeconomics consideration

In general, these issues are addressed at macroeconomic level, where long-run interest rates has not necessary the same meaning than in financial market. They are called *socially efficient or economic* interest rates, because they would be only affected by structural characteristics of the economy, and to be low-sensitive to monetary policy. Nevertheless, correct estimates of these rates are therefore useful for long term decision making, and understanding their determinants is important.

Ramsey rule and equilibrium interest rates The macroeconomics literature typically relates the *economic equilibrium* rate to the time preference rate and to the average rate of productivity growth. A typical example is the Ramsey rule proposed in the seminal paper of Ramsey [35] in 1928 where economic interest rates were linked with the marginal utility of the aggregate consumption at the economic equilibrium. More precisely, the economy is represented by the strategy of a risk-averse representative agent, whose utility function on consumption rate at date t is the function $v(t, c)$. Using an equilibrium point of view, the Ramsey rule at time 0 connects the equilibrium rate for maturity T with the marginal utility $v_c(t, c)$ of the random optimal consumption rate (c_t^*) by

$$R_0^e(T) = -\frac{1}{T} \ln \frac{\mathbb{E}[v_c(T, c_T^*)]}{v_c(0, c_0)}. \quad (5.1)$$

An usual setting is to assume separable in time utility function with exponential decay at rate $\beta > 0$ and constant risk aversion α , ($0 < \alpha < 1$), that is $v(t, c) = K e^{-\beta t} \frac{c^{1-\alpha}}{1-\alpha}$. β is the pure time preference parameter, i.e. β quantifies the agent preference of immediate goods versus future ones. The optimal consumption rate is then exogenous modelled as a geometric Brownian motion, $c_t^* = c_0 \exp((g - \frac{1}{2}\sigma^2)t + \sigma W_t)$ with g the growth rate of the economy. The Ramsey rule induces a flat curve

$$R_0^e(T) = \beta + \alpha g - \frac{1}{2}\alpha(\alpha + 1)\sigma^2. \quad (5.2)$$

The Ramsey rule is still the reference equation even if the framework in consideration is more realistic, as it was discussed by numerous economists, such as Gollier [9, 13, 8, 12, 11, 7, 10] and Weitzman [39, 40]. The equilibrium yield curve at time 0 is then computed through the Ramsey rule, using the maximum principle and leaving undiscussed the time-consistency of such an approach.

Dynamic utility functions seem to be well adapted for modeling and studying long term yield curves and their dynamics, because it allows to get rid of the dependency on the maturity T_H of the classical backward optimization problem and thus gives time consistency for the optimal choices. Besides, as dynamic utility functions take into account that the preferences and risk aversion of investor may change with time, they are also more accurate. Indeed, in the presence of generalized long term uncertainty, the decision scheme must evolve: the economists agree on the necessity of a sequential decision scheme that allows to revise the first decisions according to the evolution of the knowledge and to direct experiences, see Lecocq and Hourcade [24]. Besides, a sequential decision allows to cope with situations in which it is important to find the core of an agreement between partners having different views or anticipations, in order to give time for solving their controversy.

5.2 The financial framework

Cox-Ingersoll-Ross [3] adopt an equilibrium approach to endogenously determine the term structure of interest rates, in the presence of a financial market. In their model, there

exists a single consumption good and the production process follows a diffusion whose coefficients depends on an exogeneous stochastic factor which in some way influences the economy. The risk-free rate is determined endogenously such that the investor is not better off by trading in the money market, i.e. she is indifferent between an investment in the production opportunity and the risk-free instrument.

The financial point of view presented now is very closed to the previous one, but the agent may invest in a financial market in addition to the money market. We consider an arbitrage approach with exogenously given interest rate, instead of an equilibrium approach that determines them endogenously (see the Lecture notes of Björk [2] for a comparison between these two approaches). The financial market is an incomplete Itô financial market: notations are the one described in Section 4.1, with a n standard Brownian motion W , a (exogeneous) financial short term interest rate (r_t) and a n -dimensional risk premium (η_t^R) .

The (backward) classical optimization problem In the classical optimization problem (4.16) with given horizon T_H , studied in Subsection 4.5, both utility functions for terminal wealth and consumption rate are deterministic, then designed by small letter u and v ; their Fenchel conjugates are denoted by (\tilde{u}, \tilde{v}) .

Since we are concerned essentially by the Ramsey rule and the yield curve dynamics, we focus on the equivalent dual formulation (4.17). The optimal consumption rate $c^{*,H}(y)$ depends on the time horizon T_H through the optimal state price density process $Y^{*,H}$

$$(4.18) \begin{cases} c_t^{*,H}(c_0) = -\tilde{v}_y(t, Y_t^{*,H}(y)) & \text{i.e.} & v_c(t, c_t^{*,H}(y)) = Y_t^{*,H}(y), \quad 0 \leq t \leq T_H \\ c_0 = -\tilde{v}_y(0, y) & \text{i.e.} & v_c(0, c_0) = y \end{cases}$$

As the Lagrange multiplier y does not have an obvious financial interpretation, we adopt as in the economic literature the parameterization by the initial consumption c_0 , based on the one to one correspondance $v_c(0, c_0) = y$.

Equation (4.18) may be interpreted as a pathwise Ramsey rule, between the marginal utility of the optimal consumption and the optimal state price density process:

$$\frac{v_c(t, c_t^{*,H}(c_0))}{v_c(0, c_0)} = \frac{Y_t^{*,H}(y)}{y}, \quad 0 \leq t \leq T_H \quad \text{with} \quad v_c(0, c_0) = y. \quad (5.3)$$

The (forward point of view) dynamic problem We adopt notations of Section 4, using capital letter to refer to progressive utilities. The pathwise relation (4.18) still holds for progressive utility functions, using the characterization of the optimal consumption (see Theorem 4.7), where the parameterization is done through the initial wealth x , or equivalently c_0 or y since $c_0 = -\tilde{v}_y(u_x(x)) = -\tilde{v}_y(y)$,

$$V_c(t, c_t^*(c_0)) = Y_t^*(y), \quad t \geq 0 \quad \text{with} \quad v_c(0, c_0) = y. \quad (5.4)$$

The forward point of view emphasises the key rule played by the monotony of Y with respect to the initial condition y , under regularity conditions of the progressive utilities (cf

Theorem 4.7). Then as function of y , c_0 is decreasing, and $c_t^*(c_0)$ is an increasing function of c_0 . This question of monotony is frequently avoided, maybe because with power utility functions (the example often used in the litterature) $Y_t^*(y)$ is linear in y as ν^* does not dependent on y .

The optimal state price density process Y^* summarizes all the difference between the classical backward and dynamic forward approachs. In particular, progressive utilities allows to get rid of the dependency on the maturity T_H and thus gives time consistency for the optimal choices.

Remark: This time inconsistency is also present in the Ramsey rule (5.1) in the economic litterature. Indeed, the optimization problem they considered is usually formulated through a time separable utility $v(t, x) = e^{-\beta t}v(x)$ with a infinite horizon, which is equivalent¹ (in expectation) to consider the utility v and a random horizon τ_H exponentially distributed with parameter β . In the Ramsey rule (5.1), the optimal consumption process c^* intrinsically depends on β , which corresponds to the dependency on the horizon T_H of our classical backward formulation.

5.3 Equilibrium and financial yields curve dynamics

As previously observed, forward and backward optimization problems lead to the same pathwise relation (5.4) between optimal consumption and optimal state price density. The main difference is in the dependence on the horizon of optimal quantities in the backward case. So, in general the notation of the forward case are used, but with the additional symbol H ($Y^{*,H}, c^{*,H}, X^{*,H}$) to address the dependency on T_H in the classical backward problem.

(i) Thanks to the pathwise relation (5.4), the Ramsey rule yields to a description of the equilibrium interest rate as a function of the optimal state price density process Y^* , $R_0^e(T)(y) = -\frac{1}{T} \ln \mathbb{E}[Y_T^*(y)/y]$, that allows to give a financial interpretation in terms of zero coupon bonds. More dynamically in time,

$$R_t^e(T)(y) := -\frac{1}{T-t} \ln \mathbb{E} \left[\frac{V_c(T, c_T^*(c_0^*))}{V_c(t, c_t^*(c_0))} \middle| \mathcal{F}_t \right] = -\frac{1}{T-t} \ln \mathbb{E} \left[\frac{Y_T^*(y)}{Y_t^*(y)} \middle| \mathcal{F}_t \right] \quad \forall t < T. \quad (5.5)$$

Thanks to the flow property, $\{Y_T^*(y) = Y_T^*(Y_t^*(y)), c_T^*(c_0^*) = c_T^*(c_t^*(c_0^*)), t < T\}$, the equilibrium yield curve starting at time t with initial condition $c_t^*(c_0) = -\tilde{V}_y(t, Y_t^*(y))$ is given by $(R_t^e(T)(Y_t^*(y)), t < T)$.

(ii) The question is reduced to give a financial interpretation in terms of price of zero-coupon bonds, of the quantities $\mathbb{E} \left[\frac{Y_T^*(y)}{Y_t^*(y)} \middle| \mathcal{F}_t \right] \quad \forall t < T$. Let $(B^m(t, T), t \leq T)$, (m for market), be the price at time t of a zero-coupon bond paying one unit of cash at maturity T . In finance, the market yield curve is defined through the price of zero-coupon bond by $B^m(t, T) = \exp(-R_t^m(T)(T-t))$. We use the results of Subsection 4.6 concerning the pricing of contingent claim: the case of zero-coupon bond $B^m(t, T)$ corresponds to $\zeta_T = 1$.

¹ If τ_H is distributed as an independent exponential law with parameter β ,

$$\mathbb{E}(\int_0^{+\infty} e^{-\beta t} v(c_t) dt) = \mathbb{E}(\int_0^{\tau_H} v(c_t) dt).$$

Marginal utility yield curve (i) In a complete market, or if the zero-coupon bonds are hedgeable, $B^m(t, T)$ is computed by the the minimal risk neutral pricing rule $B^m(t, T) = \mathbb{E}[Y_{t,T}^0 | \mathcal{F}_t] = \mathbb{E}^\mathbb{Q}[e^{-\int_t^T r_s ds} | \mathcal{F}_t]$.

Then, for replicable bond, equilibrium interest rate and market interest rate coincide.

(ii) For non hedgeable zero-coupon bond, we can apply the marginal indifference pricing rule (with consumption). So we denote by $B^u(t, T)$ (u for utility) the marginal utility price at time t of a zero-coupon bond paying one cash unit at maturity T , that is $B^u(t, T) = B_t^u(T, y) = \mathbb{E}\left[\frac{Y_T^*(y)}{Y_t^*(y)} | \mathcal{F}_t\right]$. Based on the link between optimal state price density and optimal consumption, we see that

$$B_t^u(T, y) := B^u(t, T)(y) = \mathbb{E}\left[\frac{Y_T^*(y)}{Y_t^*(y)} | \mathcal{F}_t\right] = \mathbb{E}\left[\frac{V_c(T, c_T^*(c_0))}{V_c(t, c_t^*(c_0))} | \mathcal{F}_t\right]. \quad (5.6)$$

According to the Ramsey rule (5.5), equilibrium interest rates and marginal utility interest rates are the same. Nevertheless, this last curve is robust only for small trades .

The martingale property of $Y_t^*(y)B_t^u(T, y)$ yields to the following dynamics for the zero coupon bond maturing at time T with volatility vector $\Gamma_t(T, y)$

$$\frac{dB_t^u(T, y)}{B_t^u(T, y)} = r_t dt + \Gamma_t(T, y) \cdot (dW_t + (\eta_t^\mathcal{R} - \nu_t^*(y))dt). \quad (5.7)$$

Using the classical notation for exponential martingale, $\mathcal{E}_t(\theta) = \exp\left(\int_0^t \theta_s \cdot dW_s - \frac{1}{2} \int_0^t \|\theta_s\|^2 ds\right)$, the martingale $Y_t^*(y)B_t^u(T, y)$ can be written as an exponential martingale with volatility $(\nu_t^*(y) - \eta_t^\mathcal{R} + \Gamma_t(T, y))$. In particular, using that $B_T^u(T, y) = 1$,

$$Y_T^*(y) = B_0^u(T, y)\mathcal{E}_T(\nu_t^*(y) - \eta_t^\mathcal{R} + \Gamma_t(T, y)) = y e^{-\int_0^T r_s ds} \mathcal{E}_T(\nu_t^*(y) - \eta_t^\mathcal{R}).$$

Taking the logarithm gives

$$\int_0^T r_s ds = TR_0^u(T) - \int_0^T \Gamma_t(T, y) \cdot (dW_t + (\eta_t - \nu_t^*(y))dt) + \frac{1}{2} \|\Gamma_t(T, y)\|^2 dt. \quad (5.8)$$

When the family $\Gamma_t(T, y)$ is assumed to be differentiable with respect to the maturity T , we recover the classical Heath Jarrow Morton framework [14] with the following dynamics representation of the short rate

$$r_t = f_0(t, y) - \int_0^t \partial_T \Gamma_s(t, y) \cdot (dW_s + (\eta_s - \nu_s^*(y))ds) + \frac{1}{2} \partial_t \|\Gamma_s(t, y)\|^2 ds \quad (5.9)$$

with $f_0(\cdot, y)$ being the forward short rate.

- *Yield curve dynamics and infinite maturity*

The computation of the marginal utility price of zero coupon bond is then straightforward using (5.6) leading to the yield curve dynamics ($R_t^u(T, y) = -\frac{1}{T-t} \ln B_t^u(T, y)$)

$$\begin{aligned} R_t^u(T, y) &= \frac{T}{T-t} R_0^u(T, y) - \frac{1}{T-t} \int_0^t r_s ds - \int_0^t \frac{\Gamma_s(T, y)}{T-t} dW_s \\ &\quad + \int_0^t \frac{\|\Gamma_s(T, y)\|^2}{2(T-t)} ds + \int_0^t \left\langle \frac{\Gamma_s(T, y)}{T-t}, \nu_s^* - \eta_s^\mathcal{R} \right\rangle ds. \end{aligned}$$

Along the same lines as in Dybvig [33] and in El Karoui and alii. [31], we study the dynamics behavior of the yield curve for infinite maturity, when the maturity goes to infinity

$$l_t(y) := \lim_{T \rightarrow +\infty} R_t^u(T, y). \quad (5.10)$$

If $\lim_{T \rightarrow +\infty} \frac{\Gamma_t(T, y)}{T-t}$ is not equal to zero $dt \otimes d\mathbb{P}$ a.s. then $\lim_{T \rightarrow +\infty} \frac{\|\Gamma_t(T, y)\|^2}{T-t} = +\infty$ a.s and $l_t(y)$ is infinite. Otherwise, $l_t = l_0 + \int_0^t \lim_{T \rightarrow +\infty} \left(\frac{\|\Gamma_s(T, y)\|^2}{2(T-s)} \right) ds$ thus l_t is constant if $\lim_{T \rightarrow +\infty} \frac{\|\Gamma_t(T, y)\|^2}{T-t} = 0$ and l_t is a non-decreasing process if $\lim_{T \rightarrow +\infty} \frac{\|\Gamma_t(T, y)\|^2}{T-t} > 0$.

Remark 5.1. When hedging strategies cannot be implemented, the nominal amount of the transactions becomes an important risk factor and marginal utility prices are not accurate any more, especially when the market is highly illiquid. To face this problem, the utility based indifference pricing methodology seems to be more appropriate.

The *utility indifference price* is the cash amount \hat{p} for which the investor is indifferent between selling or buying a certain quantity q of a positive claim ζ_{T_H} (paid at time T_H) at the price \hat{p} in an optimally managed portfolio with initial wealth $x + p$ or investing optimally its initial wealth in the market without the claim ζ_{T_H} . If $q > 0$ (resp. $q < 0$) $-p =: p^b$ is a positive buying (resp. $p =: p^s$ is a selling) indifference price. In other words, considering the two backward maximization problems recalled in (4.22):

$$\mathcal{U}^\zeta(t, x + \hat{p}_t, q) = \mathcal{U}(t, x), \quad \text{for all } t \in [0, T_H]. \quad (5.11)$$

The pricing rule is now non linear, providing a bid-ask spread. Since it is not possible to develop this idea here, we refer the interested reader to the book "Indifference Pricing" edited by Carmona [34].

5.4 Power utilities with consumption and yield curve properties

To be able to give more precise properties of the marginal utility yield curve, we study progressive and backward power utilities as the classical most important example for economics, due to the simplification of some calculation.

Consumption consistent progressive power utility Let us consider a consumption consistent progressive power utility (with risk aversion coefficient α), associated with the pair of power progressive utilities $(U^{(\alpha)}(t, x) = \hat{Z}_t^{(\alpha)} \frac{x^{1-\alpha}}{1-\alpha}, V^{(\alpha)}(t, x) = (\hat{\psi}_t)^\alpha U^{(\alpha)}(t, x))$. From Corollary 4.6, the optimal processes are linear with respect of their initial condition, i.e. $\hat{X}_t^*(x) = x \hat{X}_t^*$, $Y_t^*(y) = y Y_t^*$, and $c_t^*(z) = z \hat{\psi}_t > 0$. The coefficient $\hat{Z}_t^{(\alpha)}$ is determined by the optimal processes via $\hat{Z}_t^{(\alpha)} = Y_t^*(\hat{X}_t^*)^\alpha$. Moreover, $d\hat{X}_t^* = \hat{X}_t^*((r_t - \hat{\psi}_t)dt + \kappa_t^*(dW_t + \eta_t^{\mathcal{R}}))$ and $dY_t^* = Y_t^*(-r_t dt + (\nu_t^* - \eta_t^{\mathcal{R}})dW_s)$ where only the dynamics of \hat{X}_t^* is affected by the consumption rate $\hat{\psi}_t$.

Power Backward Utilities and Yields curve (i) For backward utility function, the time horizon T_H plays a crucial. For zero coupon with maturity $T < T_H$, the payoff at time T is recapitalized at the risk-free rate from time T to time T_H , leading to the marginal utility price for zero-coupon (as explained in (4.29)) $B_t^{u,H}(T, y) = \mathbb{E} \left[\frac{Y_T^{*,H}(y)}{Y_t^{*,H}(y)} | \mathcal{F}_t \right]$. Since the value function of a power backward utility problem is a consistent power utility process with deterministic value at maturity T_H , the previous system (I) in Section 4.3 states that we are looking for optimal processes X^* and Y^* such that $Z_{T_H}^* = Y_{T_H}^* (X_{T_H}^*)^\alpha$ is a constant C . Thus, compared to the forward case, the dependency on the time horizon T_H adds a deterministic constraint between optimal wealth and optimal dual process at time T_H . This constraint is equivalent to the martingale property of the value function along the optimal portfolio, equal to the martingale $\frac{1}{1-\alpha} Y_t^* X_t^* = \frac{1}{1-\alpha} \mathcal{E}_t(\kappa^* - \eta^{\mathcal{R}} + \nu^*)$. To understand the impact of the short rate uncertainty, it is better to write the constraint as

$$X_{T_H}^* Y_{T_H}^0 = K Y_{T_H}^0 (Y_{T_H}^* X_{T_H}^*)^{1/1-\alpha}$$

since both processes $X^* Y^0$ and $Y^* X^*$ are exponential martingales with known volatility given respectively by $\kappa^* + \eta^{\mathcal{R}}$ and $\nu^* + \kappa^* - \eta^{\mathcal{R}}$, and $Y_t^0 = \exp(-\int_0^t r_s ds) L_t^0$ where L_t^0 is an exponential martingale with volatility $-\eta^{\mathcal{R}}$. To characterise the parameters of all these processes, we can use the uniqueness of the decomposition as terminal value of some exponential martingale, after taking into account the randomness of spot rate r or risk premia $\eta^{\mathcal{R}}$, and ν^* . In any case, this condition implies some links on the random variable $\int_0^{T_H} r_s ds$ and the volatilities $\nu_t^* - \eta_t^{\mathcal{R}}$ and κ_t^* of the optimal processes Y^* and X^* . But it is not so easy to give a description of this links in all generality.

Examples in log-normal market of marginal utility yields curves with backward power utilities

We assume a log-normal market:

- (i) $\eta^{\mathcal{R}}$ is a deterministic process (and \mathcal{R}_t contains the deterministic processes)
- (ii) $(\int_0^t r_s ds)_{0 \leq t \leq T_H}$ is a Gaussian process, with a deterministic volatility vector $\Gamma(t)$, Thus the logarithm of Y^0 is a Gaussian process and equation (5.8) can be written as

$$-\int_0^t r_s ds = \text{Cst}(t) + \int_0^t \Gamma_s(t) \cdot dW_s, \quad t \in [0, T_H]. \quad (5.12)$$

- (iii) Assuming furthermore that $\nu^{*,H}$ is deterministic, the logarithm of the optimal wealth $\ln(X^{*,H})$ and of the optimal state price density $\ln(Y^{*,H})$ are Gaussian process.

In particular, at time T_H

$$\begin{aligned} \ln(Y_{T_H}^{*,H}) &= \text{Cst} - \int_0^{T_H} r_t dt + \int_0^{T_H} (\nu_t^{*,H} - \eta_t^{\mathcal{R}}) \cdot dW_t \\ \ln(X_{T_H}^{*,H}) &= \text{Cst} + \int_0^{T_H} r_t dt + \int_0^{T_H} \kappa_t^* dW_t, \end{aligned}$$

and since $Y_{T_H}^{*,H} (X_{T_H}^{*,H})^\alpha$ is a constant, the Gaussian variable $(1-\alpha) \int_0^{T_H} \Gamma_t(T_H) \cdot dW_t + \int_0^{T_H} (\nu_t^{*,H} - \eta_t^{\mathcal{R}}) dW_t + \int_0^{T_H} \alpha \kappa_t^{*,H} dW_t$ has 0 variance. Thus, using the decomposition of

$\Gamma_t(T_H)$ into two orthogonal vectors $\Gamma_t^{\mathcal{R}}(T_H)$ and $\Gamma_t^\perp(T_H)$, we have that

$$\nu_t^{*,H} = -(1 - \alpha)\Gamma_t^\perp(T_H), \quad \alpha\kappa_t^{*,H} + (1 - \alpha)\Gamma_t^{\mathcal{R}}(T_H) = \eta_t^{\mathcal{R}}. \quad (5.13)$$

Remark that $\nu^{*,H}$ is always proportional to $\Gamma^\perp(T_H)$ and $\kappa^{*,H}$ depends on the maturity T_H only through $\Gamma^{\mathcal{R}}(T_H)$. So the knowledge of deterministic risk premium $\eta^{\mathcal{R}}$, and the optimal deterministic parameters $\nu_t^{*,H}$, $\kappa_t^{*,H}$ allows us to identify the volatility of the marginal utility zero-coupon bond with maturity T_H , as

$$\Gamma_t(T_H) = \frac{(\eta_t^{\mathcal{R}} - \nu_t^{*,H})}{(1 - \alpha)} - \frac{\alpha}{1 - \alpha}\kappa_t^{*,H} \quad (5.14)$$

Conversely, given a deterministic volatility for the zero-coupon bond with maturity T_H , and the risk aversion coefficient α , we can easily recover from Equation (5.13) the optimal volatilities $\nu_t^{*,H}$ and $\kappa_t^{*,H}$.

A classical model for the short rate dynamics is the Vasicek model, where the short rate is given by an Ornstein-Uhlenbeck process $dr_t = a(b - r_t)dt - \sigma dW_t$. The computation of $\int_0^t r_s ds$ yields the volatility for the zero-coupon bond $\Gamma_s(t) = (1 - e^{-a(t-s)})\frac{\sigma}{a}$ (see for example [25] Proposition 2.6.1.6 for details of this classical Gaussian computation). The classical framework consists in a complete market driven by a one dimensional Brownian motion. In the framework of an incomplete market with the noise driving the spot rate being orthogonal to the one driving the risky assets, then $\Gamma_s^\perp(t) = (1 - e^{-a(t-s)})\frac{\sigma}{a}$ and $\Gamma^{\mathcal{R}} = 0$. Thus in this example $\kappa_t^{*,H} = \frac{\eta_t^{\mathcal{R}}}{\alpha}$ does not depend on the maturity T_H while $\nu_t^{*,H} = (\alpha - 1)(1 - e^{-a(T_H-t)})\frac{\sigma}{a}$ depends on the time to maturity $(T_H - t)$.

- *Yield curve for infinite maturity* Since in the backward case $\nu^{*,H}$ depends on the maturity T_H , the yield curve for infinite maturity $l_t = \lim_{T \rightarrow +\infty} R_t^u(T)$ differs from the one in the forward case (in (5.10)) if $\lim_{T \rightarrow +\infty} \frac{\|\Gamma_t(T)\|^2}{T-t} > 0$ and $\alpha < \frac{1}{2}$. As we are looking at the asymptotics $T \rightarrow +\infty$ and $T \leq T_H$, we set $T_H = T \rightarrow +\infty$ (note that similar results hold if $T_H > T \rightarrow +\infty$) and

$$\begin{aligned} l_t &= l_0 + \int_0^t \lim_{T \rightarrow +\infty} \left(\frac{\|\Gamma_s(T)\|^2}{2(T-s)} - (1 - \alpha) \frac{\|\Gamma_s^\perp(T)\|^2}{(T-s)} \right) ds \\ l_t &= l_0 + \int_0^t \lim_{T \rightarrow +\infty} \left(\frac{(2\alpha - 1)\|\Gamma_s^\perp(T)\|^2}{2(T-s)} + \frac{\|\Gamma_s^{\mathcal{R}}(T)\|^2}{2(T-s)} \right) ds \end{aligned}$$

Thus if $\lim_{T \rightarrow +\infty} \frac{\|\Gamma_t(T)\|^2}{T-t} > 0$, l_t is a non-decreasing function of the risk aversion α : if $\alpha \geq \frac{1}{2}$, l_t is a non-decreasing process as in the forward case; if $\alpha < 1/2$, l_t may be decreasing or increasing, depending on the sign of $\lim_{T \rightarrow +\infty} \left(\frac{(2\alpha-1)\|\Gamma_s^\perp(T)\|^2}{2(T-s)} + \frac{\|\Gamma_s^{\mathcal{R}}(T)\|^2}{2(T-s)} \right)$. In particular, $\lim_{T \rightarrow +\infty} \frac{\|\Gamma_t(T)\|^2}{T-t} > 0$, $\alpha < \frac{1}{2}$ and $\lim_{T \rightarrow +\infty} \frac{\|\Gamma_t^{\mathcal{R}}(T)\|^2}{2(T-t)} = 0$ implies a decreasing yield curve for infinite maturity in this framework of backward power utilities in log-normal market.

An affine factor model makes it possible to extend the previous log-normal model to a more stochastic framework, while leading to tractable pricing formulas, see [22].

Conclusion: In this paper, we remained deliberately closed to the economic setting, studying more precisely power utility functions and using marginal utility indifference price (Davis price) for the pricing of non replicable zero-coupon bonds, which allowed us to interpret the Ramsey rule in a financial framework. Those simplifications imply that the impact of the initial economic wealth is avoided : on the one hand power utilities imply that the optimal processes are linear with respect to the initial conditions, on the other hand Davis price is a linear pricing rule while for non replicable claim, the size of the transaction is an important source of risk that must be taken into account. This important issue concerning the dependency on the initial wealth and its impact for yield curves will be discussed in a future work.

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